



## DEPARTMENT OF MATHEMATICS

### UNIT-I FOURIER SERIES

#### FOURIER SERIES:

A periodic function  $f(x)$  which satisfies certain condition can be expressed as cosine and sine series of the form,

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

is called Fourier series of  $f(x)$  and  $a_0, a_n, b_n (n=1, 2, \dots)$  are called Fourier coefficients of  $f(x)$ .

Obtain the Fourier series expansion for the function  $f(x) = x^2$  with period  $2\pi$ , in the interval  $0 < x < 2\pi$

Soln:

$$\text{Let } f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx.$$

$$\text{Now } a_0 = \frac{1}{\pi} \int_0^{2\pi} f(x) dx$$

$$= \frac{1}{\pi} \int_0^{2\pi} x^2 dx = \frac{1}{\pi} \left[ \frac{x^3}{3} \right]_0^{2\pi} = \frac{1}{3\pi} \times 8\pi^3 = \frac{8\pi^2}{3}$$

$$\therefore a_0 = \frac{8\pi^2}{3}$$

$$a_n = \frac{1}{\pi} \int_0^{2\pi} f(x) \cos nx dx$$

$$= \frac{1}{\pi} \int_0^{2\pi} x^2 \cos nx dx$$



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$$\begin{array}{l}
 u = x^2 \quad v = \cos nx \\
 u' = 2x \quad v_1 = \frac{\sin nx}{n} \\
 u'' = 2 \quad v_2 = -\frac{\cos nx}{n^2} \\
 u''' = 0 \quad v_3 = -\frac{\sin nx}{n^3}
 \end{array}$$

$$= \frac{1}{\pi} \left[ x^2 \frac{\sin nx}{n} - 2x \left( -\frac{\cos nx}{n^2} \right) + 2 \left( -\frac{\sin nx}{n^3} \right) \right]_0^{2\pi}$$

$$= \frac{1}{\pi} \left[ 2x \frac{\cos nx}{n^2} \right]_0^{2\pi}$$

$$= \frac{2}{\pi n^2} [2\pi \cos n(2\pi) - 0]$$

$$= \frac{2}{\pi n^2} \cdot 2\pi = \frac{4}{n^2}$$

$$b_n = \frac{1}{\pi} \int_0^{2\pi} f(x) \sin nx \, dx$$

$$= \frac{1}{\pi} \int_0^{2\pi} x^2 \sin nx \, dx$$

$$\begin{array}{l}
 u = x^2 \quad v = \sin nx \\
 u' = 2x \quad v_1 = -\frac{\cos nx}{n} \\
 u'' = 2 \quad v_2 = -\frac{\sin nx}{n^2} \\
 u''' = 0 \quad v_3 = \frac{\cos nx}{n^3}
 \end{array}$$



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$$\begin{aligned}
&= \frac{1}{\pi} \left[ n^2 \left( -\frac{\cos nm}{n} \right) - 2n \left( -\frac{\sin nm}{n^2} \right) + 2 \left( \frac{\cos nm}{n^3} \right) \right]_0^{2\pi} \\
&= \frac{1}{\pi} \left[ -\frac{n^2 \cos n\pi}{n} + \frac{2 \cos n\pi}{n^3} \right]_0^{2\pi} \\
&= \frac{1}{\pi} \left[ -\frac{4\pi^2 \cos n(2\pi)}{n} + \frac{2 \cos n(2\pi)}{n^3} - \left( 0 + \frac{2 \cos n(0)}{n^3} \right) \right] \\
&= \frac{1}{\pi} \left[ -\frac{4\pi^2}{n} + \frac{2}{n^3} - \frac{2}{n^3} \right]
\end{aligned}$$

$$b_n = -\frac{4\pi}{n}$$

$$\therefore f(x) = \frac{4\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4}{n^2} \cos nx - \sum_{n=1}^{\infty} \frac{4\pi}{n} \sin nx$$

Obtain the Fourier series for  $f(x) = \frac{(\pi-x)^2}{4}$  in  $(0, 2\pi)$

$$\begin{aligned}
a_0 &= \frac{1}{\pi} \int_0^{2\pi} \frac{(\pi-x)^2}{4} dx = \frac{1}{\pi} \int_0^{2\pi} \frac{(\pi-x)^2}{4} dx \\
&= \frac{1}{\pi} \left[ \frac{(\pi-x)^3}{4 \cdot 3} (-1) \right]_0^{2\pi} = \frac{-1}{\pi} \left[ \frac{(-\pi)^3}{6} - \frac{\pi^3}{6} \right] = \frac{-1}{\pi} \left[ \frac{-\pi^3}{6} - \frac{\pi^3}{6} \right] = \frac{-1}{\pi} \left[ \frac{-2\pi^3}{6} \right] = \frac{2\pi^2}{6} = \frac{\pi^2}{3}
\end{aligned}$$

$$a_n = \frac{1}{\pi} \int_0^{2\pi} \frac{(\pi-x)^2}{4} \cos nx dx$$

$$a_0 = \frac{\pi^2}{3}$$



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$$= \frac{1}{4\pi} \left[ (\pi-x)^2 \frac{\sin nx}{n} + 2(\pi-x) \left( \frac{\cos nx}{n^2} \right) + 2(-1) \left( \frac{-\sin nx}{n^3} \right) \right]_0^{2\pi} \quad (10)$$

$$= \frac{1}{4\pi} \left[ \pi^2 \frac{\sin n2\pi}{n} + 2(-\pi) \left( \frac{-\cos n(2\pi)}{n^2} \right) + 2 \frac{\sin n(2\pi)}{n^3} - \left[ 2(\pi) \left( \frac{-\cos n(0)}{n^2} \right) \right] \right]$$

$$= \frac{1}{4\pi} \left[ 2\pi \frac{(-1)^n}{n^2} + 2 \frac{\pi}{n^2} \right] = \frac{1}{n^2}$$

$$b_n = \frac{1}{4\pi} \int_0^{2\pi} (\pi-x)^2 \sin nx \, dx$$

$$= \frac{1}{4\pi} \left[ \frac{(\pi-x)^3}{3} \left( \frac{\cos nx}{n} \right) - 2(\pi-x)(-1) \left( \frac{-\sin nx}{n^2} \right) + 2(-1) \left( \frac{\cos nx}{n^3} \right) \right]_0^{2\pi}$$

$$= \frac{1}{4\pi} \left[ -\pi^2 \frac{\cos n(2\pi)}{n} - 2(-\pi) \frac{\sin n(2\pi)}{n^2} - 2 \frac{\cos n(2\pi)}{n^3} - \left[ -\pi^2 \frac{\cos n(0)}{n} - 2 \frac{\cos n(0)}{n^3} \right] \right]$$



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$$= \frac{1}{4\pi} \left[ -\frac{\pi^2}{n} - \frac{2}{n^3} + \frac{\pi^2}{n} + \frac{2}{n^3} \right] = 0$$

$$f(x) = \frac{\pi^2}{12} + \sum_{n=1}^{\infty} \frac{1}{n^2} \cos n x$$

$$f(x) = \frac{\pi^2}{12} + \left[ \frac{\cos x}{1} + \frac{\cos 2x}{2^2} + \dots \right]$$

$$\begin{aligned} b_n &= \frac{1}{l} \int_0^{2l} (l-x)^2 \sin\left(\frac{n\pi}{l}x\right) dx \\ &= \frac{1}{l} \left[ (l-x)^2 \left( -\frac{\cos\left(\frac{n\pi}{l}x\right)}{\left(\frac{n\pi}{l}\right)} \right) - 2(l-x)(-1) \left( -\frac{\sin\left(\frac{n\pi}{l}x\right)}{\left(\frac{n\pi}{l}\right)^2} \right) + \right. \\ &\quad \left. 2 \left( \frac{\cos\left(\frac{n\pi}{l}x\right)}{\left(\frac{n\pi}{l}\right)^3} \right) \right]_{0}^{2l} \end{aligned}$$