FOURIER SERIES:

A periodic function $f(x)$ which satisfies certain condition can be expressed as cosine and sine series of the form,

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

is called Fourier series of $f(x)$ and $a_0, a_n, b_n\ (n=1,2,..)$ are called Fourier co-efficients of $f(x)$.

Obtain the Fourier series expansion for the function $f(x) = x^2$ with period 2π, in the interval $0 < x < 2\pi$

Soln:

Let $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$.

Now $a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx$

$$= \frac{1}{\pi}\int_0^{2\pi} x^2\,dx = \frac{1}{\pi}\left[\frac{x^3}{3}\right]_0^{2\pi} = \frac{1}{3\pi} \times 8\pi^3 = \frac{8\pi^2}{3}$$

$\therefore\ a_0 = \frac{8\pi^2}{3}$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx$$

$$= \frac{1}{\pi}\int_0^{2\pi} x^2 \cos nx\,dx.$$

$u = x^2 \qquad v = \cos nx$

$u' = 2x \qquad v_1 = \sin nx/n$

$u'' = 2 \qquad v_2 = -\cos nx/n^2$

$u''' = 0 \qquad v_3 = -\sin nx/n^3$

$$= \frac{1}{\pi}\left[x^2 \frac{\sin \cancel{nx}}{\cancel{n}} - 2x\left(-\frac{\cos nx}{n^2}\right) + 2\left(-\frac{\sin \cancel{nx}}{\cancel{n^3}}\right)\right]_0^{2\pi}$$

$$= \frac{1}{\pi}\left[2x\frac{\cos nx}{n^2}\right]_0^{2\pi}$$

$$= \frac{2}{\pi n^2}\left[2\pi \cos n(2\pi) - 0\right]$$

$$= \frac{2}{\cancel{\pi} n^2} \cdot 2\cancel{\pi} = \frac{4}{n^2}$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x) \sin nx \, dx.$$

$$= \frac{1}{\pi}\int_0^{2\pi} x^2 \sin nx \, dx.$$

$u = x^2 \qquad v = \sin nx$

$u' = 2x \qquad v_1 = -\cos nx/n$

$u'' = 2 \qquad v_2 = -\sin nx/n^2$

$u''' = 0 \qquad v_3 = \cos nx/n^3$

$$= \frac{1}{\pi}\left[x^2\left(-\frac{\cos nx}{n}\right) - 2x\left(-\frac{\cancel{\sin nx}}{n^2}\right) + 2\left(\frac{\cos nx}{n^3}\right)\right]_0^{2\pi}$$

$$= \frac{1}{\pi}\left[-\frac{x^2\cos nx}{n} + \frac{2\cos nx}{n^3}\right]_0^{2\pi}$$

$$= \frac{1}{\pi}\left[-\frac{4\pi^2\cos n(2\pi)}{n} + \frac{2\cos n(2\pi)}{n^3} - \left(0 + \frac{2\cos n(0)}{n^3}\right)\right]$$

$$= \frac{1}{\pi}\left[-\frac{4\pi^2}{n} + \frac{2}{n^3} - \frac{2}{n^3}\right]$$

$$b_n = -\frac{4\pi}{n}$$

$$\therefore f(x) = \frac{4\pi^2}{3} + \sum_{n=1}^{\infty}\frac{4}{n^2}\cos nx - \sum_{n=1}^{\infty}\frac{4\pi}{n}\sin nx$$

Obtain the Fourier series for $f(x) = \left(\frac{\pi - x}{2}\right)^2$ in $(0, 2\pi)$

$$a_0 = \frac{1}{\pi}\int_0^{2\pi}\frac{(\pi-x)^2}{4}\,dx = \frac{1}{\pi}\int_0^{2\pi}\frac{(\pi-x)^2}{4}\,dx$$

$$= \frac{1}{\pi}\left[\frac{(\pi-x)^3}{4\cdot 3}(-1)\right]_0^{2\pi} = \frac{-1}{\pi}\,\frac{(-\pi)^3 - \pi^3}{6\cdot 2} = \frac{2\pi^2}{6\cdot 2} = \frac{\pi^2}{6}$$

$-\frac{1}{12\pi}((-\pi)^3 - (\pi^3))$

$$\boxed{a_0 = \frac{\pi^2}{6}}$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi}\frac{(\pi-x)^2}{4}\cos nx\,dx$$

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$$= \frac{1}{4\pi}\left[(\pi-x)^2\frac{\sin nx}{n} + 2(\pi-x)\left(\frac{-\cos nx}{n^2}\right) + 2(-1)\left(-\frac{\sin nx}{n^3}\right)\right]_0^{2\pi}$$

$$= \frac{1}{4\pi}\left[\pi^2\frac{\sin n2\pi}{n} + 2(-\pi)\left(-\frac{\cos n(2\pi)}{n^2}\right) + 2\frac{\sin n(2\pi)}{n^3} - \left[2(\pi)\left(-\frac{\cos n(0)}{n^2}\right)\right]\right]$$

$$= \frac{1}{4\pi}\left[\frac{2\pi(-1)^n}{n^2} + \frac{2\pi}{n^2}\right] = \frac{1}{n^2}$$

$$b_n = \frac{1}{4\pi}\int_0^{2\pi}(\pi-x)^2 \sin nx \, dx$$

$$= \frac{1}{4\pi}\left[(\pi-x)^2\left(-\frac{\cos nx}{n}\right) - 2(\pi-x)(-1)\left(-\frac{\sin nx}{n^2}\right) + 2(-1)\left(+\frac{\cos nx}{n^3}\right)\right]_0^{2\pi}$$

$$= \frac{1}{4\pi}\left[-\pi^2\frac{\cos n(2\pi)}{n} - 2(-\pi)\frac{\sin n(2\pi)}{n^2} - 2\frac{\cos n(2\pi)}{n^3} - \left[-\pi^2\frac{\cos n(0)}{n} - 2\frac{\cos n(0)}{n^3}\right]\right]$$

$$= \frac{1}{4\pi}\left[-\frac{\pi^2}{n} - \frac{2}{n^3} + \frac{\pi^2}{n} + \frac{2}{n^3} \right] = 0$$

$$f(x) = \frac{\pi^2}{12} + \sum_{n=1}^{\infty} \frac{1}{n^2} \cos nx$$

$$= \frac{\pi^2}{12} + \left[\frac{\cos x}{1} + \frac{\cos 2x}{2^2} + \cdots \right]$$

$$b_n = \frac{1}{l} \int_0^{2l} (l-x)^2 \sin\left(\frac{n\pi}{l}\right)x \, dx$$

$$= \frac{1}{l}\left[(l-x)^2 \left(-\frac{\cos(n\pi/l)x}{(n\pi/l)} \right) - 2(l-x)(-1)\left(-\frac{\sin(n\pi/l)x}{(n\pi/l)^2} \right) + 2\left(\frac{\cos(n\pi/l)x}{(n\pi/l)^3} \right) \right]_0^{2l}$$