CHANGE OF INTERVAL :

Fourier series Expansion in $(0, 2l)$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi}{l}x + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi}{l}x$$

where
$$a_0 = \frac{1}{l}\int_c^{c+2l} f(x)\,dx$$

$$a_n = \frac{1}{l}\int_c^{c+2l} f(x)\cos\frac{n\pi}{l}x\,dx$$

$$b_n = \frac{1}{l}\int_c^{c+2l} f(x)\sin\frac{n\pi}{l}x\,dx.$$

1) Find the Fourier series expansion of period $2l$ for the function $f(x) = (l-x)^2$ in $(0, 2l)$

Soln:

Let $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi}{l}x + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi}{l}x$.

Now $a_0 = \frac{1}{l}\int_0^{2l}(l-x)^2\,dx$

$$= \frac{1}{l}\left[\frac{(l-x)^3}{-3}\right]_0^{2l} = \frac{1}{l}\left[\frac{-l^3 - l^3}{-3}\right]$$

$$= \frac{2l^2}{3}$$

$$a_n = \frac{1}{l}\int_0^{2l}(l-x)^2\cos\frac{n\pi x}{l}\,dx$$

$$= \frac{1}{l}\left[(l-x)^2\cancel{\frac{\sin\frac{n\pi x}{l}}{\frac{n\pi}{l}}} - 2(l-x)(-1)\left(-\frac{\cos\frac{n\pi}{l}x}{\left(\frac{n\pi}{l}\right)^2}\right) + 2\left(-\frac{\sin\frac{n\pi x}{l}}{\left(\frac{n\pi}{l}\right)^3}\right)\right]_0^{2l}$$

$$= \frac{1}{l}\left[-2(l-x)\frac{\cos\left(\frac{n\pi}{l}\right)x}{\left(\frac{n\pi}{l}\right)^2}\right]_0^{2l}$$

$$= \frac{-2}{l}\cdot\left(\frac{l}{n\pi}\right)^2\left[(-l)\cos(2n\pi) - (l)\cos 0\right]$$

$$= -\frac{2}{l}\times\frac{l^2}{(n\pi)^2}\left[-l-l\right]$$

$$= 4\frac{l^2}{n^2\pi^2}$$

$$b_n = \frac{1}{l}\int_0^{2l}(l-x)^2\sin\left(\frac{n\pi}{l}\right)x\,dx.$$

$$= \frac{1}{l}\left[(l-x)^2\left(-\frac{\cos(n\pi/l)x}{(n\pi/l)}\right) - 2(l-x)(-1)\left(-\frac{\sin(n\pi/l)x}{(n\pi/l)^2}\right) + 2\left(\frac{\cos(n\pi/l)x}{(n\pi/l)^3}\right)\right]_0^{2l}$$

$$= \frac{1}{l}\left[-l^2 \frac{\cos n\pi}{(n\pi/l)} + 2\frac{\cos 2n\pi}{(n\pi/l)^3} - \left(l^2 \frac{\cos 0}{(n\pi/l)} + 2\frac{\cos 0}{(n\pi/l)^3}\right)\right]$$

$$= \frac{1}{l}\left[-l^2 \cdot \frac{l}{n\pi} + 2\frac{l^3}{(n\pi)^3} + l^2 \cdot \frac{l}{n\pi} - 2 \cdot \frac{l^3}{(n\pi)^3}\right]$$

$$= \frac{1}{l}[0]$$

$$= 0$$

$$\therefore f(x) = \frac{l^2}{3} + \frac{4l^2}{\pi^2} \sum_{n=1}^{\infty} \frac{\cos(n\pi/l)x}{n^2}.$$

2) Obtain the Fourier series expansion for $f(x) = \begin{cases} x, & 0<x<1 \\ 1-x, & 1<x<2 \end{cases}$

Soln:

Here the interval is $(0, 2)$.

Length $= b-a$

$2l = 2 \Rightarrow l = 1$

$$\therefore\ f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos n\pi x + \sum_{n=1}^{\infty} b_n \sin n\pi x .$$

Now $a_0 = \frac{1}{1}\int_0^2 f(x)\,dx$

$$= \int_0^1 x\,dx + \int_1^2 (1-x)\,dx$$

$$= \left[\frac{x^2}{2}\right]_0^1 + \left[x - \frac{x^2}{2}\right]_1^2$$

$$= \frac{1}{2} + \left[\left(2 - \frac{4}{2}\right) - \left(1 - \frac{1}{2}\right)\right]$$

$$= \frac{1}{2} + \left[0 - \frac{1}{2}\right]$$

$$= 0$$

$$a_n = \frac{1}{1}\int_0^2 f(x) \cos n\pi x\,dx$$

$$= \int_0^1 x \cos n\pi x\,dx + \int_1^2 (1-x) \cos n\pi x\,dx$$

$$= \left[x\frac{\sin n\pi x}{n\pi} - 1\cdot\left(-\frac{\cos n\pi x}{(n\pi)^2}\right)\right]_0^1 +$$

$$\left[(1-x)\frac{\sin n\pi x}{n\pi} - (-1)\left(-\frac{\cos n\pi x}{(n\pi)^2}\right)\right]_1^2$$

$$= \left[\frac{\cos n\pi}{(n\pi)^2} - \frac{\cos 0}{(n\pi)^2}\right] + \left[-\frac{\cos 2n\pi}{(n\pi)^2} - \left(-\frac{\cos n\pi}{(n\pi)^2}\right)\right]$$

$$= \frac{(-1)^n}{(n\pi)^2} - \frac{1}{(n\pi)^2} - \frac{1}{(n\pi)^2} + \frac{(-1)^n}{(n\pi)^2}$$

$$= \left[(-1)^n - 1\right]\frac{2}{(n\pi)^2}.$$

$$= \frac{2\left[(-1)^n - 1\right]}{n^2\pi^2}$$

$$b_n = \frac{1}{1}\int_0^2 f(x)\sin n\pi x\,dx$$

$$= \int_0^1 x\sin n\pi x\,dx + \int_1^2 (1-x)\sin n\pi x\,dx.$$

$$= \left[x\left(-\frac{\cos n\pi x}{n\pi}\right) - 1\left(-\frac{\cancel{\sin n\pi x}}{(n\pi)^2}\right)\right]_0^1 + \left[(1-x)\left(-\frac{\cos n\pi x}{n\pi}\right) - (-1)\left(-\frac{\cancel{\sin n\pi x}}{(n\pi)^2}\right)\right]_1^2$$

$$= \left[-\frac{\cos n\pi}{n\pi} - 0\right] + \left[(-1)\cdot -\frac{\cos 2n\pi}{n\pi} - 0\right]$$

$$= -\frac{(-1)^n}{n\pi} + \frac{1}{n\pi}$$

$$= \frac{1-(-1)^n}{n\pi}.$$

$$\therefore f(x) = \sum_{n=1}^{\infty} \frac{2\left[(-1)^n - 1\right]}{n^2\pi^2}\cos n\pi x + \sum_{n=1}^{\infty}\left[\frac{1-(-1)^n}{n\pi}\right]\sin n\pi x.$$