EVEN AND ODD FUNCTION:

A real function 'f' is called an even function if $f(-x) = f(x)$ for all x. If $f(x)$ is an even function then $\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx$. Eg: x^2, $\cos x$, $|x|$, $x \sin x$.

A real function 'f' is called an odd function if $f(-x) = -f(x)$ for all x. If $f(x)$ is an odd function then $\int_{-a}^{a} f(x)\,dx = 0$. Eg: x^3, $\sin x$, $x \cos x$.

FOURIER SERIES EXPANSION IN $(-\pi, \pi)$: $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$

where $a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx$; $a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx$; $b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx$

If $f(x)$ is an odd function, then Fourier expansion is

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx, \text{ where } b_n = \frac{2}{\pi}\int_{0}^{\pi} f(x)\sin nx\,dx.$$

[$a_0 = a_n = 0$, since $f(x)$ is an odd function]

If $f(x)$ is an even function, then Fourier expansion is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx, \text{ where } a_0 = \frac{2}{\pi}\int_{0}^{\pi} f(x)\,dx$$

$$\& \quad a_n = \frac{2}{\pi}\int_{0}^{\pi} f(x)\cos nx\,dx$$

[$b_n = 0$, since $f(x)$ even. $\Rightarrow$ even $\times$ odd $= 0$]

1) Find the Fourier Series for $f(x) = x^2$ in $-\pi \le x \le \pi$ and deduce that

(i) $\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\cdots = \frac{\pi^2}{6}$

(ii) $\frac{1}{1^2}-\frac{1}{2^2}+\frac{1}{3^2}-\cdots = \frac{\pi^2}{12}$

(iii) $\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\cdots = \frac{\pi^2}{8}$

Soln: $f(x) = x^2$ is an even function.

Let $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$.

Now
$$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx$$
$$= \frac{2}{\pi}\int_0^{\pi} x^2\,dx = \frac{2}{\pi}\left[\frac{x^3}{3}\right]_0^{\pi}$$
$$= \frac{2\pi^2}{3}$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx.$$
$$= \frac{2}{\pi}\int_0^{\pi} x^2 \cos nx\,dx$$
$$= \frac{2}{\pi}\left[x^2\frac{\cancel{\sin nx}}{n} - 2x\left(\frac{-\cos nx}{n^2}\right) + 2\left(-\frac{\cancel{\sin nx}}{n^3}\right)\right]_0^{\pi}$$
$$= \frac{2}{\pi}\left[\frac{2x\cos nx}{n^2}\right]_0^{\pi}$$
$$= \frac{2}{\cancel{\pi}}\cdot\frac{2\cancel{\pi}\cos n\pi}{n^2}$$
$$= \frac{4(-1)^n}{n^2}$$

$$\therefore f(x) = \frac{\pi^2}{3} + \sum_{n=1}^{\infty}\frac{4(-1)^n}{n^2}\cos nx.$$

Deduction:

$$f(x) = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos nx$$

put $x = \pi$ is the disconuity & end point.

$$\frac{f(\pi) + f(-\pi)}{2} = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos n\pi$$

$$\frac{\pi^2 + \pi^2}{2} = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cdot (-1)^n$$

$$\pi^2 - \frac{\pi^2}{3} = 4\sum_{n=1}^{\infty} \frac{(-1)^{2n}}{n^2}$$

$$\frac{\cancel{2}\pi^2}{3 \times \cancel{4}_2} : \frac{\pi^2}{6} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots$$

(ii) put $x = 0$ is continuous point.

$$\therefore f(0) = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos n(0)$$

$$0 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$$

$$\frac{\pi^2}{3} = -4\left[-\frac{1}{1^2} + \frac{1}{2^2} - \frac{1}{3^2} + \frac{1}{4^2} - \cdots\right]$$

$$\frac{\pi^2}{12} = \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots$$

(iii) By adding (i) & (ii), we get

$$\frac{\pi^2}{6}+\frac{\pi^2}{12} = \frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\cdots+\frac{1}{1^2}-\frac{1}{2^2}+\frac{1}{3^2}-\frac{1}{4^2}+\cdots$$

$$\frac{\cancel{3}\pi^2}{\cancel{12}_4} = 2\left[\frac{1}{1^2}+\frac{1}{3^2}+\cdots\right]$$

$$\frac{\pi^2}{8} = \frac{1}{1^2}+\frac{1}{3^2}+\cdots$$

> $f(x) = x(\pi^2-x^2)$ in $(-\pi,\pi)$.

Soln: $f(x)$ is an odd function.

Let $f(x) = \sum_{n=1}^{\infty} b_n \sin nx$

Now $b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\, dx$

$$= \frac{2}{\pi}\int_0^{\pi} x(\pi^2-x^2)\sin nx\, dx.$$

$$= \frac{2}{\pi}\int_0^{\pi}(x\pi^2-x^3)\sin nx\, dx$$

$$= \frac{2}{\pi}\left[(x\pi^2 - x^3)\left(-\frac{\cos nx}{n}\right) - (\pi^2 - 3x^2)\left(-\frac{\cancel{\sin nx}}{n^2}\right) + (-6x)\left(\frac{\cos nx}{n^3}\right) - (-6)\left(\frac{\cancel{\sin nx}}{n^4}\right)\right]_0^\pi$$

$$= \frac{2}{\pi}\left[-6\pi \frac{\cos n\pi}{n^3}\right]$$

$$= -\frac{12(-1)^n}{n^3}$$

$$b_n = \frac{12(-1)^{n+1}}{n^3}$$

$$\therefore f(x) = 12\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^3} \sin nx .$$

4) $f(x) = |x|$ in $-\pi < x < \pi$

Soln: $f(x)$ is an even function

Let $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$

Now $a_0 = \frac{2}{\pi}\int_0^\pi f(x)\,dx$

$$= \frac{2}{\pi}\int_0^\pi x\,dx = \frac{2}{\pi}\left[\frac{x^2}{2}\right]_0^\pi = \pi$$

$a_0 = \pi$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx$$

$$= \frac{2}{\pi}\int_0^{\pi} x\cos nx\,dx$$

$$= \frac{2}{\pi}\left[x\left(\frac{\cancel{\sin nx}}{n}\right) - 1\left(-\frac{\cos nx}{n^2}\right)\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[\frac{\cos nx}{n^2}\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[\frac{(-1)^n}{n^2} - \frac{1}{n^2}\right]$$

$$f(x) = \frac{\pi}{2} + \sum_{n=1}^{\infty}\frac{2}{\pi}\left[\frac{(-1)^n - 1}{n^2}\right]\cos nx.$$

$$f(x) = \begin{cases} 1 + \frac{2x}{\pi}, & -\pi < x < 0 \\ 1 - \frac{2x}{\pi}, & 0 < x < \pi \end{cases}$$

NOTE:
$\phi_1(x) = \phi_2(x)$, even
$\phi_1(x) = -\phi_2(x)$, odd.

Soln:

$$f(-x) = \begin{cases} 1 - \frac{2x}{\pi} \\ 1 + \frac{2x}{\pi} \end{cases} = f(x), \text{ is an even function.}$$

Let $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$.

Now $a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx$

$= \frac{2}{\pi}\int_0^{\pi}\left(1-\frac{2}{\pi}x\right)dx$

$= \frac{2}{\pi}\left[x - \frac{2}{\pi}\frac{x^2}{2}\right]_0^{\pi}$

$= \frac{2}{\pi}\left[\pi - \pi\right]$

$= 0$

$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx$

$= \frac{2}{\pi}\int_0^{\pi}\left(1-\frac{2}{\pi}x\right)\cos nx\,dx$

$= \frac{2}{\pi}\left[\left(1-\frac{2}{\pi}x\right)\left(\cancel{\frac{\sin nx}{n}}\right) - \left(-\frac{2}{\pi}\right)\left(-\frac{\cos nx}{n^2}\right)\right]_0^{\pi}$

$= \frac{2}{\pi}\left[-\frac{2}{\pi}\frac{\cos n\pi}{n^2} - \left(-\frac{2}{\pi}\frac{\cos 0}{n^2}\right)\right]$

$= -\frac{4}{\pi^2 n^2}\left[(-1)^n - 1\right] = \frac{4}{\pi^2 n^2}\left[1-(-1)^n\right]$

$\therefore\ f(x) = \sum_{n=1}^{\infty} \frac{4}{\pi^2 n^2}\left[1-(-1)^n\right]\cos nx.$