Find the fourier transform of the function $f(x) = \begin{cases} 1-|x|, & |x|<1 \\ 0, & |x|>1 \end{cases}$ and deduce that $\int_0^\infty \left(\frac{\sin t}{t}\right)^2 dt$ & $\int_0^\infty \left(\frac{\sin t}{t}\right)^4 dt$.

$$F(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} (1-|x|) e^{isx} dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} (1-|x|)(\cos sx + i \sin sx)\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} (1-|x|) \cos sx\, dx + \frac{i}{\sqrt{2\pi}} \int_{-1}^{1} (1-|x|) \sin sx\, dx$$

Since the second term is an odd function, it becomes zero.

$$\therefore F(s) = \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} (1-|x|) \cos sx\, dx$$

$$= \frac{2}{\sqrt{2\pi}} \int_{0}^{1} (1-|x|) \cos sx\, dx$$

$$= \sqrt{\frac{2}{\pi}} \left[(1-x) \frac{\sin sx}{s} - (-1)\left(-\frac{\cos sx}{s^2}\right) \right]_0^1$$

$$= \sqrt{\frac{2}{\pi}} \left[-\frac{\cos s}{s^2} - \left[-\frac{1}{s^2}\right] \right] = \sqrt{\frac{2}{\pi}} \left[\frac{1-\cos s}{s^2} \right]$$

$1 - \cos s = 2 \sin^2 s/2$

$1 - 2\sin^2\theta = \cos 2\theta$

$1 - 2\sin^2\theta/2 = \cos\theta$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{2 \sin^2 s/2}{s^2} \right]$$

Inversion fourier transform:

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \sqrt{\frac{2}{\pi}} \cdot \frac{2 \sin^2 s/2}{s^2} e^{-isx} ds.$$

$$1-x = \frac{2}{\pi} \int_{-\infty}^{\infty} \frac{\sin^2 s/2}{s^2} [\cos sx - i \sin sx]\, ds$$

$$1-x = \frac{2}{\pi} \int_{-\infty}^{\infty} \frac{\sin^2 s/2}{s^2} \cos sx\, ds$$ [$\because$ the second integral term is zero, (odd func.)]

$$1-x = \frac{4}{\pi} \int_{0}^{\infty} \frac{\sin^2 s/2}{s^2} \cos sx\, ds$$

put $x = 0$,

$$1 = \frac{4}{\pi} \int_{0}^{\infty} \frac{\sin^2 s/2}{s^2} ds.$$

Take $s/2 = t \Rightarrow s = 2t$.
$ds = 2dt$.

$$\Rightarrow 1 = \frac{4}{\pi} \int_0^{\infty} \frac{\sin^2 t}{(2t)^2} \cdot 2dt. = \frac{4}{\pi} \int_0^{\infty} \frac{\sin^2 t}{\cancel{4}t^2} \cancel{2}dt$$

$$1 = \frac{2}{\pi} \int_0^{\infty} \frac{\sin^2 t}{t^2} dt.$$

$$\Rightarrow \int_0^{\infty} \frac{\sin^2 t}{t^2} dt = \pi/2. \Rightarrow \int_0^{\infty} \left(\frac{\sin t}{t}\right)^2 dt = \pi/2$$

Parseval's Identity:

$$\int_{-\infty}^{\infty} (f(x))^2 dx = \int_{-\infty}^{\infty} (F(s))^2 ds$$

$$\int_{-1}^{1} (1-x)^2 dx = \int_{-\infty}^{\infty} \left[\sqrt{\frac{2}{\pi}} \left[\frac{2\sin^2 s/2}{s^2}\right]\right]^2 ds$$

$$2\frac{(1-x)^3}{-3}\Big|_0^1 = \frac{2\times 4}{\pi} \int_{-\infty}^{\infty} \left(\frac{\sin^2 s/2}{s^2}\right)^2 ds$$

$$\frac{2}{3} = \frac{8\times 2}{\pi} \int_0^{\infty} \left(\frac{\sin^2 s/2}{s^2}\right)^2 ds$$

Take $s/2 = t \Rightarrow s = 2t$

$ds = 2dt$

$$\therefore \frac{2}{3} = \frac{16}{\pi} \int_0^{\infty} \left(\frac{\sin^2 t}{(2t)^2}\right)^2 2dt$$

$$= \frac{16}{\pi} \int_0^{\infty} \left(\frac{\sin^2 t}{4t^2}\right)^2 2dt$$

$$\Rightarrow \frac{2}{3} = \frac{16}{\pi}\cdot\frac{1}{16} \int_0^{\infty} \left(\frac{\sin^2 t}{t^2}\right)^2 2dt$$

$$\Rightarrow \int_0^{\infty} \left(\frac{\sin t}{t}\right)^4 dt = \pi/3$$

Parseval's Identity:

$$\int_{-\infty}^{\infty} (f(x))^2 dx = \int_{-\infty}^{\infty} (F(s))^2 ds$$

$$\int_{-1}^{1} (1-x)^2 dx = \int_{-\infty}^{\infty} \left[\sqrt{\frac{2}{\pi}}\left[\frac{2\sin^2 s/2}{s^2}\right]\right]^2 ds$$

$$2\frac{(1-x)^3}{-3}\Big|_0^1 = \frac{2\times 4}{\pi}\int_{-\infty}^{\infty}\left(\frac{\sin^2 s/2}{s^2}\right)^2 ds$$

$$\frac{2}{3} = \frac{8\times 2}{\pi}\int_0^{\infty}\left(\frac{\sin^2 s/2}{s^2}\right)^2 ds$$

Take $s/2 = t \Rightarrow s = 2t$

$ds = 2dt$

$$\therefore \frac{2}{3} = \frac{16}{\pi}\int_0^{\infty}\left(\frac{\sin^2 t}{(2t)^2}\right)^2 2dt$$

$$= \frac{16}{\pi}\int_0^{\infty}\left(\frac{\sin^2 t}{4t^2}\right)^2 2dt$$

$$\Rightarrow \frac{2}{3} = \frac{16}{\pi}\cdot\frac{1}{16}\int_0^{\infty}\left(\frac{\sin^2 t}{t^2}\right)^2 2dt$$

$$\Rightarrow \int_0^{\infty}\left(\frac{\sin t}{t}\right)^4 dt = \pi/3$$

2) $f(x) = \begin{cases} a-|x|, & |x| \le a \\ 0, & |x| > a \end{cases}$ find the fourier transform and deduce that $\int_0^{\infty}\left(\frac{\sin t}{t}\right)^2 dt$ & $\int_0^{\infty}\left(\frac{\sin t}{t}\right)^4 dt$.