1) Find the sine transform of the function $f(x) = \frac{e^{-ax}}{x}$.

Soln:

WKT $F_s(s) = F_s[f(x)] = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \sin sx \, dx$

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{e^{-ax}}{x} \sin sx \, dx.$$

D.w.r.to 's' OBS we get,

$$\frac{d}{ds}[F_s(s)] = \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{d}{ds}\left[\frac{e^{-ax}}{x} \sin sx\right] dx$$

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{e^{-ax}}{\cancel{x}} \cos sx \cdot \cancel{x} \, dx$$

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-ax} \cos sx \, dx$$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{a}{a^2+s^2}\right]$$

Integrating w.r.to 's'. we get,

$$F_s(s) = \sqrt{\frac{2}{\pi}} \int \frac{a}{a^2+s^2} \, ds$$

$$= \sqrt{\frac{2}{\pi}} \, a \int \frac{1}{a^2+s^2} \, ds$$

$$= \sqrt{\frac{2}{\pi}} \, \cancel{a} \cdot \frac{1}{\cancel{a}} \tan^{-1}\left(\frac{s}{a}\right) + c$$

$$= \sqrt{\frac{2}{\pi}} \tan^{-1}\left(\frac{s}{a}\right) + c$$

2) Find the cosine transform of the function $f(x) = \frac{e^{-ax}}{x}$.

Soln: WKT $F_c(s) = F_c[f(x)] = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \cos sx \, dx$

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{e^{-ax}}{x} \cos sx \, dx$$

D.w.r. to 's' OBS we get,

$$\frac{d}{ds}[F_c(s)] = \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{d}{ds}\left[\frac{e^{-ax}}{x} \cdot \cos sx \, dx\right]$$

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{e^{-ax}}{\cancel{x}} \cdot -\sin sx \cdot \cancel{x} \, dx$$

$$= -\sqrt{\frac{2}{\pi}} \int_0^\infty e^{-ax} \sin sx \, dx$$

$$\frac{d}{ds}[F_c(s)] = -\sqrt{\frac{2}{\pi}} \left[\frac{s}{s^2+a^2}\right]$$

Integrating w.r. to 's' we get.

$$F_c(s) = -\sqrt{\frac{2}{\pi}} \int \frac{s}{s^2+a^2} \, ds$$

$$= -\sqrt{\frac{2}{\pi}} \cdot \frac{1}{2} \log(s^2+a^2)$$

$$= -\frac{1}{\sqrt{2\pi}} \log(s^2+a^2)$$