Solve: $x(z^2-y^2)\frac{\partial z}{\partial x} + y(x^2-z^2)\frac{\partial z}{\partial y} = z(y^2-x^2)$

$\Rightarrow x(z^2-y^2)p + y(x^2-z^2)q = z(y^2-x^2)$

$P = x(z^2-y^2)$; $Q = y(x^2-z^2)$; $R = z(y^2-x^2)$

$$\frac{dx}{x(z^2-y^2)} = \frac{dy}{y(x^2-z^2)} = \frac{dz}{z(y^2-x^2)}$$

Choose first set of multiplies $(l, m, n) = (x, y, z)$

$$\Rightarrow \frac{x\,dx + y\,dy + z\,dz}{x^2(z^2-y^2) + y^2(z^2-x^2) + z^2(y^2-x^2)} = k$$

$\Rightarrow Dr = 0$

$\Rightarrow Nr: x\,dx + y\,dy + z\,dz = 0$

$\Rightarrow u(x,y,z): x^2+y^2+z^2 = c_1$

Choose second set of multiplies $(l', m', n') = (1/x, 1/y, 1/z)$

$$\Rightarrow \frac{\frac{1}{x}dx + \frac{1}{y}dy + \frac{1}{z}dz}{\frac{1}{x}x(z^2-y^2) + \frac{1}{y}(x^2-z^2) + \frac{1}{z}(y^2-x^2)} = k.$$

6, 7, 11, 14, 15, 16, 20, 24, 36, 44

$\Rightarrow Dr = 0$

$\Rightarrow Nr: \frac{1}{x}dx + \frac{1}{y}dy + \frac{1}{z}dz = 0$

$\Rightarrow v(x,y,z) = \log xyz = \log c_2$

$xyz = c_2$.

$\therefore$ General soln. is $\phi(x^2+y^2+z^2, xyz) = 0$.

Solve: $(3z-4y)p+(4x-2z)q=2y-3x$.

General form: $$\frac{dx}{3z-4y}=\frac{dy}{4x-2z}=\frac{dz}{2y-3x}.$$

Choose first set of multiplier $(l,m,n)=(x,y,z)$

$$\Rightarrow \frac{x\,dx+y\,dy+z\,dz}{x(3z-4y)+y(4x-2z)+z(2y-3x)}=k.$$

$\Rightarrow$ Dr $=0$.

$\Rightarrow$ Nr: $x\,dx+y\,dy+z\,dz=0$

$$\Rightarrow u(x,y,z)=x^2+y^2+z^2=c_1$$

Consider the other multiplier $(l',m',n')=(2,3,4)$.

$$\Rightarrow \frac{2dx+3dy+4dz}{2(3z-4y)+3(4x-2z)+4(2y-3x)}=k$$

$\Rightarrow$ Dr $=0$

$\Rightarrow$ Nr: $2dx+3dy+4dz=0$

$\Rightarrow v(x,y)=2x+3y+4z=0$

$\therefore$ G.S. $\phi(x^2+y^2+z^2,\ 2x+3y+4z)=0$

Solve: $x^2(z-y)p + y^2(x-z)q = z^2(y-x)$

General form: $\dfrac{dx}{x^2(z-y)} = \dfrac{dy}{y^2(x-z)} = \dfrac{dz}{z^2(y-x)}$

Consider the first set of multiplier $(l, m, n) = \left(\frac{1}{x^2}, \frac{1}{y^2}, \frac{1}{z^2}\right)$

$$\frac{\frac{1}{x^2}dx + \frac{1}{y^2}dy + dz\frac{1}{z^2}}{z-y+x-z+y-x} = k$$

$\Rightarrow Dr = 0$

$\Rightarrow Nr: \dfrac{dx}{x^2} + \dfrac{dy}{y^2} + \dfrac{dz}{z^2} = 0$

$\Rightarrow u(x,y,z) = \cancel{1/x} + \cancel{1/y} + 1/z = c_1$

Consider the other multiplier $(l', m', n') = \left(\frac{1}{x}, \frac{1}{y}, \frac{1}{z}\right)$

$$\frac{\frac{1}{x}dx + \frac{1}{y}dy + \frac{1}{z}dz}{x(z-y) + y(x-z) + z(y-x)} = k$$

$\Rightarrow Dr = 0$

$\Rightarrow Nr: \frac{1}{x}dx + \frac{1}{y}dy + \frac{1}{z}dz = 0$

$\Rightarrow v(x,y,z) = \log(xyz) = \log c_2.$

$$xyz = c_2$$

$\therefore$ soln is, $\phi\left(1/x + 1/y + 1/z,\ xyz\right) = 0$