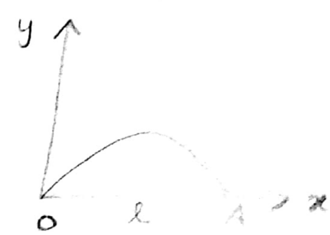
3 (8)

Problems on Vibrating String with Zero initial velocity :

(1) A uniform string is stretched (extended) and fastened (attached) to two points $x = 0$ and $x = l$ apart. Motion is started by displacing the string into the form of the curve $y = kx(l-x)$ & then released from this position at time $t=0$. Derive the expression for the displacement of any point on the string at a distance 'x' from one end at time t.

Solution :



Step 1 : The wave equation is,

$$\frac{\partial^2 y}{\partial t^2} = a^2 \frac{\partial^2 y}{\partial x^2}$$

Step 2 :

B.C (i) : $y(0,t) = 0$ for all $t > 0$

B.C (ii) : $y(l,t) = 0$ for all $t > 0$

I.C (iii) : $\left(\frac{\partial y}{\partial t}\right)_{(x,0)} = 0, \ 0 < x < l$ ($\because$ initial velocity is zero)

I.C (iv) : $y(x,0) = kx(l-x), \ 0<x<l$ (initial displacement)

Step 3 : The suitable solution which satisfies our boundary conditions are given by,

$$y(x,t) = (c_1 \cos px + c_2 \sin px)(c_3 \cos pat + c_4 \sin pat) \longrightarrow (1)$$

Step 4 : Applying condition (i) in eqn (1), we get

$$y(0,t) = (c_1 + 0)(c_3 \cos pat + c_4 \sin pat) = 0.$$

Here $(c_3 \cos pat + c_4 \sin pat) \neq 0$, since it is defined for all $t > 0$.

$$\therefore \boxed{c_1 = 0}$$

Subs $c_1 = 0$ in eqn (1), we get,

$$y(x,t) = c_2 \sin px \,(c_3 \cos pat + c_4 \sin pat) \longrightarrow (2)$$

Step 5:

Applying condition (ii) in eqn (2), we get

$$y(l,t) = c_2 \sin pl \,(c_3 \cos pat + c_4 \sin pat) = 0.$$

Here $(c_3 \cos pat + c_4 \sin pat) \neq 0$, since it is defined for all $t > 0$.

Therefore either $c_2 = 0$ or $\sin pl = 0$.

If we take $c_2 = 0$ we get a trivial solution.

$\therefore$ Take $\sin pl = 0$.

$$\sin pl = \sin n\pi \quad (\because \sin n\pi = 0)$$

$$pl = n\pi$$

$$\boxed{p = \frac{n\pi}{l}}, \text{ where } n \text{ is an integer.}$$

Substituting $p = \frac{n\pi}{l}$ in eqn (2), we get

$$y(x,t) = c_2 \sin \frac{n\pi x}{l} \left(c_3 \cos \frac{n\pi at}{l} + c_4 \sin \frac{n\pi at}{l} \right) \longrightarrow (3)$$

Step 6:

Diff (3) w.r.t 't' partially, we get,

$$\left(\frac{\partial y}{\partial t}\right)_{(x,t)} = c_2 \sin \frac{n\pi x}{l} \left(-c_3 \left(\frac{n\pi a}{l}\right) \sin \frac{n\pi at}{l} + c_4 \left(\frac{n\pi a}{l}\right) \cos \frac{n\pi at}{l} \right)$$

Now applying condition (iii), we get

$$\left(\frac{\partial y}{\partial t}\right)_{(x,0)} = c_2 \sin \frac{n\pi x}{l} \left(0 + c_4 \left(\frac{n\pi a}{l}\right)\right) = 0$$

$$\Rightarrow c_2 c_4 \left(\frac{n\pi a}{l}\right) \sin \frac{n\pi x}{l} = 0.$$

Here $c_2 \neq 0$, $\sin \frac{n\pi x}{l} \neq 0 \because$ It is defined for all x and $\frac{n\pi a}{l} \neq 0$ since all are constants.

(4)

$\therefore \boxed{C_4 = 0}$

subs $C_4 = 0$ in eqn (3), we get

$$y(x,t) = C_2 C_3 \sin\frac{n\pi x}{l} \cos\frac{n\pi a t}{l}$$

$$y(x,t) = C_n \sin\frac{n\pi x}{l} \cos\frac{n\pi a t}{l} \rightarrow (4)$$

where $C_n = C_2 C_3$.

Step 7: The most general solution is,

$$y(x,t) = \sum_{n=1}^{\infty} C_n \sin\frac{n\pi x}{l} \cos\frac{n\pi a t}{l} \rightarrow (5)$$

Step 8:

Applying B.C (iv) in eqn (5), we have

$$y(x,0) = \sum_{n=1}^{\infty} C_n \sin\frac{n\pi x}{l} = kx(l-x) \rightarrow (6)$$

Step 9:

To find C_n:

Expand $kx(l-x)$ in a half range Fourier sine series in the interval $(0,l)$.

$$kx(l-x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l} \rightarrow (7) \text{ where}$$

$$b_n = \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}\,dx$$

From (6) & (7) $b_n = C_n$

$$\therefore C_n = \frac{2}{l}\int_0^l kx(l-x)\sin\frac{n\pi x}{l}\,dx$$

$$\therefore C_n = \frac{2}{l}\left\{-(klx-kx^2)\frac{l}{n\pi}\cos\frac{n\pi x}{l} + (kl-2kx)\frac{l^2}{n^2\pi^2}\sin\frac{n\pi x}{l} - 2k\cdot\frac{l^3}{n^3\pi^3}\cos\frac{n\pi x}{l}\right\}_0^l$$

$u = klx - kx^2$

$u' = kl - 2kx$

$u'' = -2k$

$v = \sin\frac{n\pi x}{l}$

$v_1 = -\cos\frac{n\pi x}{l} \Big/ \frac{n\pi}{l}$

$v_2 = -\sin\frac{n\pi x}{l} \Big/ \frac{n^2\pi^2}{l^2}$

$v_3 = \cos\frac{n\pi x}{l} \Big/ \frac{n^3\pi^3}{l^3}$

$$C_n = \frac{2}{l}\left\{-2k\frac{l^3}{n^3\pi^3}\cos\frac{n\pi x}{l}\right\}_0^l$$

$$= -\frac{4kl^2}{n^3\pi^3}\left\{\cos n\pi - \cos 0\right\}$$

$$C_n = \frac{4kl^2}{n^3\pi^3}\left[1-(-1)^n\right]$$

$$\boxed{C_n = \begin{cases} 0 & , \text{ if } n \text{ is even} \\ \dfrac{8kl^2}{n^3\pi^3} & , \text{ if } n \text{ is odd} \end{cases}}$$

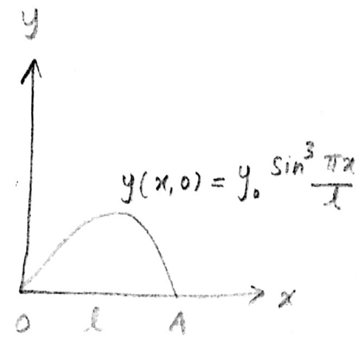
Step 10: subs the value of C_n in eqn (5), we get

$$y(x,t) = \sum_{n=odd}^{\infty} \frac{8kl^2}{n^3\pi^3} \sin\frac{n\pi x}{l}\cos\frac{n\pi at}{l}$$

$$y(x,t) = \frac{8kl^2}{\pi^3}\sum_{n=odd}^{\infty}\frac{1}{n^3}\sin\frac{n\pi x}{l}\cos\frac{n\pi at}{l}$$

(3) A tightly stretched string with fixed end points $x=0$ and $x=l$ is initially in a position given by $y(x,0) = y_0 \sin^3 \frac{\pi x}{l}$. If it is released from rest from this position find the displacement y at any distance x from one end at any time t.

Solution :



Step 1: The wave equation is

$$\frac{\partial^2 y}{\partial t^2} = a^2 \frac{\partial^2 y}{\partial x^2}$$

Step 2: The boundary conditions are

(i) $y(0,t) = 0$ for all $t > 0$

(ii) $y(l,t) = 0$ for all $t > 0$

(iii) $\left(\frac{\partial y}{\partial t}\right)_{(x,0)} = 0$, $0 < x < l$ (initial velocity is zero)

(iv) $y(x,0) = y_0 \sin^3 \frac{\pi x}{l}$, $0 < x < l$.

Step 3: The suitable solution which satisfies our boundary conditions are given by,

$$y(x,t) = (c_1 \cos px + c_2 \sin px)(c_3 \cos pat + c_4 \sin pat) \longrightarrow (1)$$

Step 4: Applying condition (i) in eqn (1), we get,

$$y(0,t) = (c_1 \cos p\cdot 0 + c_2 \sin p\cdot 0)(c_3 \cos pat + c_4 \sin pat) = 0$$

$$c_1 (c_3 \cos pat + c_4 \sin pat) = 0$$

Here $c_3 \cos pat + c_4 \sin pat \neq 0$, since it is defined for all $t > 0$.

$$\therefore \boxed{c_1 = 0}$$

Subs $c_1 = 0$ in eqn (1), we get,

$$y(x,t) = c_2 \sin px \,(c_3 \cos pat + c_4 \sin pat) \longrightarrow (2)$$

Step 5 :

Applying condition (ii) in eqn ②, we get,

$$y(l,t) = c_2 \sin pl \,(c_3 \cos pat + c_4 \sin pat) = 0.$$

Here $(c_3 \cos pat + c_4 \sin pat) \neq 0$, since it is defined for all $t > 0$.

Therefore either $c_2 = 0$ or $\sin pl = 0$.

If we take $c_2 = 0$ we get a trivial solution.

Take $\sin pl = 0$.

$$\sin pl = \sin n\pi \quad (\because \sin n\pi = 0)$$

$$pl = n\pi$$

$$\boxed{p = \frac{n\pi}{l}}, \text{ where } n \text{ is an integer.}$$

Subs $p = \frac{n\pi}{l}$ in eqn ②, we get,

$$y(x,t) = c_2 \sin \frac{n\pi x}{l} \left(c_3 \cos \frac{n\pi a t}{l} + c_4 \sin \frac{n\pi a t}{l} \right) \rightarrow ③$$

Step 6 :

Diff ③ w.r.t 't' partially, we get.

$$\left(\frac{\partial y}{\partial t}\right) = c_2 \sin \frac{n\pi x}{l} \left(-c_3 \left(\frac{n\pi a}{l}\right) \sin \frac{n\pi a t}{l} + c_4 \left(\frac{n\pi a}{l}\right) \cos \frac{n\pi a t}{l} \right)$$

Now applying condition (iii), we get

$$\left(\frac{\partial y}{\partial t}\right)_{(x,0)} = c_2 \sin \frac{n\pi x}{l} \left(0 + c_4 \left(\frac{n\pi a}{l}\right)\right) = 0$$

$$\Rightarrow c_2 c_4 \left(\frac{n\pi a}{l}\right) \sin \frac{n\pi x}{l} = 0$$

Here $c_2 \neq 0$, $\sin \frac{n\pi x}{l} \neq 0$ $\because$ it is defined for all x

and $\frac{n\pi a}{l} \neq 0$ since all are constants.

$$\therefore \boxed{c_4 = 0}$$

Subs $c_4 = 0$ in eqn ③, we get

(13)

$$y(x,t) = c_2 c_3 \sin\frac{n\pi x}{l} \cos\frac{n\pi at}{l}$$

$$y(x,t) = c_n \sin\frac{n\pi x}{l} \cos\frac{n\pi at}{l} \rightarrow (4)$$

where $c_n = c_2 c_3$.

Step 7:

The most general solution is,

$$y(x,t) = \sum_{n=1}^{\infty} c_n \sin\frac{n\pi x}{l} \cos\frac{n\pi at}{l} \rightarrow (5)$$

Step 8:

Now applying condition (iv) in (5),

$$y(x,0) = \sum_{n=1}^{\infty} c_n \sin\frac{n\pi x}{l} = y_0 \sin^3\frac{\pi x}{l}$$

$$\sum_{n=1}^{\infty} c_n \sin\frac{n\pi x}{l} = \frac{y_0}{4}\left(3\sin\frac{\pi x}{l} - \sin\frac{3\pi x}{l}\right)$$

$$\left[\because \sin^3 x = \frac{1}{4}(3\sin x - \sin 3x)\right]$$

$$\therefore c_1 \sin\frac{\pi x}{l} + c_2 \sin\frac{2\pi x}{l} + c_3 \sin\frac{3\pi x}{l} + \cdots = \frac{y_0}{4}\left(3\sin\frac{\pi x}{l} - \sin\frac{3\pi x}{l}\right)$$

Equating like coefficients on either side, we have

$$c_1 = \frac{3y_0}{4},\ c_2 = 0,\ c_3 = -\frac{y_0}{4},\ c_4 = 0,\ c_5 = 0, \ldots$$

$\therefore$ Eqn (5) gives

$$y(x,t) = c_1 \sin\frac{\pi x}{l}\cos\frac{\pi at}{l} + c_2 \sin\frac{2\pi x}{l}\cos\frac{2\pi at}{l} + c_3 \sin\frac{3\pi x}{l}\cos\frac{3\pi at}{l} + \cdots \rightarrow (6)$$

Step 9:

Subs the values of $c_1, c_2, c_3, \ldots$ in (6),

$$y(x,t) = \frac{3y_0}{4}\sin\frac{\pi x}{l}\cos\frac{\pi at}{l} - \frac{y_0}{4}\sin\frac{3\pi x}{l}\cos\frac{3\pi at}{l}$$

Note: $\sin^3\frac{\pi x}{l}$ contains only two terms. Hence we need not expand $\sin^3\frac{\pi x}{l}$ in a half range sine series.