Problem:

1. Find the Fourier transform of $f(x) = \begin{cases} x, & |x| < a \\ 0, & |x| > a \end{cases}$

Solution:

Here $f(x) = x$ in $-a < x < a$

$f(x) = 0$ in $-\infty < x < -a$ and $a < x < \infty$

$$F[f(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{isx} dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} x e^{isx} dx = \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} x \left(\cos(sx) + i \sin(sx) \right) dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} \left(x \cos(sx) + x i \sin(sx) \right) dx$$

$$= \frac{1}{\sqrt{2\pi}} \left(\int_{-a}^{a} x \cos sx \, dx + \int_{-a}^{a} x i \sin(sx) \, dx \right)$$

$$= \frac{1}{\sqrt{2\pi}}\left(0 + 2\int_0^a x\, i\sin(sx)\,dx\right)$$

$x \cos x \rightarrow$ odd
$O \times E \longrightarrow$

$x \sin x \rightarrow$ even
$O \times O \longrightarrow$

$$= \frac{2i}{\sqrt{2\pi}} \int_0^a x\sin(sx)\,dx$$

$$= i\sqrt{\frac{2}{\pi}}\left[x\left(\frac{-1}{s}\cos(sx)\right) + \frac{1}{s^2}\sin(sx)\right]_0^a$$

$$= i\sqrt{\frac{2}{\pi}}\left[-\frac{a}{s}\cos(as) + \frac{1}{s^2}(\sin(as)) - (0)\right]$$

$$= i\sqrt{\frac{2}{\pi}}\,\frac{1}{s^2}\left[\sin(as) - as\cos(as)\right]$$

$$F[f(x)] = F(s) = i\sqrt{\frac{2}{\pi}}\left[\frac{\sin(as) - as\cos(as)}{s^2}\right]$$

2) Find the Fourier Transform of $f(x) = \begin{cases} 1, & |x| < a \\ 0, & |x| > a \end{cases}$

Solution:

Here $f(x) = 1 \quad , \quad -a < x < a$

$f(x) = 0 \quad , \quad -\infty < x < -a$ and $a < x < \infty$

$$F[f(x)] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)\, e^{isx}\,dx$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-a}^{a} 1\, e^{isx}\,dx = \frac{1}{\sqrt{2\pi}}\int_{-a}^{a} e^{isx}\,dx$$

$e^{i\theta}$ is neither odd nor even

$$= \frac{1}{\sqrt{2\pi}}\left[\frac{e^{isx}}{is}\right]_{-a}^{a}$$

$$= \frac{1}{\sqrt{2\pi}}\,\frac{1}{is}\left[e^{is(a)} - e^{is(-a)}\right]$$

$$= \frac{1}{\sqrt{2\pi}}\,\frac{1}{is}\left[e^{ias} - e^{-ias}\right]$$

$$\frac{e^{ix} - e^{-ix}}{2i} = \sin x$$

$$= \frac{1}{\sqrt{2\pi}}\,\frac{1}{s}\left[\frac{e^{ias} - e^{-ias}}{i}\right] = \frac{1}{\sqrt{2\pi}}\,\frac{1}{s}\left[2\sin as\right]$$

$$= \sqrt{\frac{2}{\pi}}\left[\frac{\sin as}{s}\right] \qquad \boxed{F(s) = \sqrt{\frac{2}{\pi}}\left[\frac{\sin as}{s}\right]}$$