Eigen values and Eigen vectors of a real matrix

1. Find the eigen values and eigen vectors of $\begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix}$

Solution:

To find the characteristics Equation and eigen values

Let $A = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix}$

The characteristic equation is given by $|A - \lambda I| = 0$

$$\begin{vmatrix} 2-\lambda & 1 & 0 \\ 0 & 2-\lambda & 1 \\ 0 & 0 & 2-\lambda \end{vmatrix} = 0$$

$$(2-\lambda)(2-\lambda)(2-\lambda) = 0$$

$$\lambda = 2,2,2$$

The eigen values are 2,2,2.

To find the eigen vectors

The eigen vector $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ is given by $(A - \lambda I)X = 0$

$$\begin{bmatrix} 2-\lambda & 1 & 0 \\ 0 & 2-\lambda & 1 \\ 0 & 0 & 2-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\left.\begin{aligned} (2-\lambda)x_1 + x_2 + 0x_3 &= 0 \\ 0x_1 + (2-\lambda)x_2 + x_3 &= 0 \\ 0x_1 + 0x_2 + (2-\lambda)x_3 &= 0 \end{aligned}\right\} (1)$$

Case(i)

When $\lambda = 2$, the system of equations (1) becomes

$$0x_1 + x_2 + 0x_3 = 0 \rightarrow (2)$$

$$0x_1 + 0x_2 + x_3 = 0 \rightarrow (3)$$

$$0x_1 + 0x_2 + 0x_3 = 0 \rightarrow (4)$$

Taking equation (2) and (3), Applying cross rule method we get

x_1 $\quad$ x_2 $\quad$ x_3

1 $\quad$ 0 $\quad$ 0 $\quad$ 1

0 $\quad$ 1 $\quad$ 0 $\quad$ 0

$$\frac{x_1}{1} = \frac{x_2}{0} = \frac{x_3}{0}$$

The eigen vector is $X_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

Since the second and third eigen vectors are same,

we have $X_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ and $X_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

Characteristic Equation	Eigen Values	Eigen Vectors
$(2 - \lambda)^3 = 0$	$\lambda_1 = 2$	$X_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$
	$\lambda_2 = 2$	$X_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

	$\lambda_3 = 2$	$X_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

PROBLEMS ON SYMMETRIC MATRICES WITH DIFFERENT EIGEN VALUES

1. Find the eigen values and eigen vectors of $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & -1 \\ 0 & -1 & 3 \end{bmatrix}$

Solution:

To find the characteristics Equation and eigen values

Let $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & -1 \\ 0 & -1 & 3 \end{bmatrix}$

The characteristic equation is given by $|A - \lambda I| = 0$

$$\begin{vmatrix} 1-\lambda & 0 & 0 \\ 0 & 3-\lambda & -1 \\ 0 & -1 & 3-\lambda \end{vmatrix} = 0$$

$$(1-\lambda)[(3-\lambda)^2 - 1] = 0$$

$1 - \lambda = 0 \quad , \qquad [(3-\lambda)^2 - 1] = 0$

$\lambda = 1 \qquad , \qquad 9 + \lambda^2 - 6\lambda - 1 = 0$

$$\lambda^2 - 6\lambda + 8 = 0$$

$$\lambda = 4, \qquad \lambda = 2$$

The eigen values are 1,2,4.

To find the eigen vectors

The eigen vector $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ is given by $(A - \lambda I)X = 0$

$$\begin{bmatrix} 1-\lambda & 0 & 0 \\ 0 & 3-\lambda & -1 \\ 0 & -1 & 3-\lambda \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\left.\begin{aligned} (1-\lambda)x_1 + 0x_2 + 0x_3 &= 0 \\ 0x_1 + (3-\lambda)x_2 - x_3 &= 0 \\ 0x_1 - x_2 + (3-\lambda)x_3 &= 0 \end{aligned}\right\} (1)$$

Case(i)

When $\lambda = 1$, the system of equations (1) becomes

$$0x_1 + 0x_2 + 0x_3 = 0 \rightarrow (2)$$

$$0x_1 + 2x_2 - x_3 = 0 \rightarrow (3)$$

$$0x_1 - x_2 + 2x_3 = 0 \rightarrow (4)$$

Taking equation (2) and (3), Applying cross rule method we get

$x_1 x_2$ x_3

2 -1 0 2

-12 0 -1

$$\frac{x_1}{3} = \frac{x_2}{0} = \frac{x_3}{0}$$

The eigen vector is $X_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

Case(ii)

When $\lambda = 2$, the system of equations (1) becomes

$$-x_1 + 0x_2 + 0x_3 = 0 \rightarrow (5)$$

$$0x_1 + x_2 - x_3 = 0 \rightarrow (6)$$

$$0x_1 - x_2 + x_3 = 0 \rightarrow (7)$$

Taking equation (5) and (6), Applying cross rule method we get

$x_1 x_2$ x_3

0	0	-1	0
1	-1	0	1

$$\frac{x_1}{0} = \frac{x_2}{-1} = \frac{x_3}{-1}$$

The eigen vector is $X_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$

Case(iii)

When $\lambda = 4$, the system of equations (1) becomes

$$-3x_1 + 0x_2 + 0x_3 = 0 \rightarrow (8)$$

$$0x_1 - x_2 - x_3 = 0 \rightarrow (9)$$

$$0x_1 - x_2 - x_3 = 0 \rightarrow (10)$$

Taking equation (8) and (9), Applying cross rule method we get

x_1 x_2 x_3

0	0	-3	0
-1	-1	0	-1

$$\frac{x_1}{0} = \frac{x_2}{-3} = \frac{x_3}{3}$$

The eigen vector is $X_3 = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$

Characteristic Equation	Eigen Values	Eigen Vectors
$(2-\lambda)^3 = 0$	$\lambda_1 = 1$	$X_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$
	$\lambda_2 = 2$	$X_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$
	$\lambda_3 = 4$	$X_3 = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$

PROBLEMS ON SYMMETRIC MATRICES WITH REPEATED EIGEN VALUES

1. Find the eigen values and eigen vectors of $\begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$

Solution:

To find the characteristics Equation and eigen values

Let $A = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$

The Characteristic equation is given by

$$\lambda^3 - c_1\lambda^2 + c_2\lambda - c_3 = 0$$

Where

c_1= Sum of leading diagonal elements

$$= 2 + 2 + 2$$

$$= 6$$

c_2=Sum of the minors of leading diagonal elements.

$$= \begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix} + \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} + \begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix}$$

$$= 3 + 3 + 3$$

$$= 9$$

c_3=det[A]

$$= \begin{vmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{vmatrix}$$

$= 2(4 - 1) + 1(-2 + 1) + 1(1 - 2)$

$= 6 - 1 - 1$

$= 4$

Sub the values of a_1, a_2, a_3 in characteristic equation

$$\lambda^3 - c_1\lambda^2 + c_2\lambda - c_3 = 0$$

We get $\lambda^3 - 6\lambda^2 + 9\lambda - 4 = 0$

To find eigen values

$\lambda = 1$is a root.

	1	-6	9	-4
0	1	-5	4	
	1	-5	4	0

$\lambda^2 - 5\lambda + 4 = 0$

$\lambda = 4, \ \ \lambda = 1$

The eigen values are 1,1,4.

To find the eigen vectors

The eigen vector $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ is given by $(A - \lambda I)X = 0$

$$\begin{bmatrix} 2-\lambda & -1 & 1 \\ -1 & 2-\lambda & -1 \\ 1 & -1 & 2-\lambda \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\left.\begin{aligned} (2-\lambda)x_1 - x_2 + x_3 &= 0 \\ -x_1 + (2-\lambda)x_2 - x_3 &= 0 \\ x_1 - x_2 + (2-\lambda)x_3 &= 0 \end{aligned}\right\} (1)$$

Case(i)

When $\lambda = 4$, the system of equations (1) becomes

$$-2x_1 - x_2 + x_3 = 0 \rightarrow (2)$$

$$-x_1 - 2x_2 - x_3 = 0 \rightarrow (3)$$

$$x_1 - x_2 - 2x_3 = 0 \rightarrow (4)$$

Taking equation (3) and (4), Applying cross rule method we get

x_1 x_2 x_3

-1 1 -2 -1

-2 -1 -1 -2

$$\frac{x_1}{3} = \frac{x_2}{-3} = \frac{x_3}{3}$$

The eigen vector is $X_1 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$

Case(ii)

When $\lambda = 1$, the system of equations (1) becomes

$$x_1 - x_2 + x_3 = 0 \rightarrow (2)$$

$$-x_1 + 2x_2 - x_3 = 0 \rightarrow (3)$$

$$x_1 - x_2 + x_3 = 0 \rightarrow (4)$$

The above equations are same

Hence we take any one of the above equation as $x_1 = x_2 - x_3$

Put $x_2 = k_1, x_3 = k_2$

We have $X_2 = \begin{bmatrix} k_1 - k_2 \\ k_1 \\ k_2 \end{bmatrix}$

The simplest Eigen vector is obtained by putting $k_1 = 1, k_2 = 0$

We get $X_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$

The eigen vector is $X_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$

The other Eigen vector for $\lambda = 1$ is obtained by putting $k_1 = 0, k_2 = -1$

We get $X_3 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$

Characteristic Equation	Eigen Values	Eigen Vectors
	$\lambda_1 = 4$	$X_1 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$

$\lambda^3 - 6\lambda^2 + 9\lambda - 4 = 0$	$\lambda_2 = 1$	$X_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$
	$\lambda_3 = 1$	$X_2 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$