



UNIT 3- DIFFERENTIAL CALCULUS

Centre of Curvature

Centre of Curvature:

The centre of curvature of a curve $y = f(x)$ at a point $P(x, y)$ is denoted by $C(\bar{x}, \bar{y})$ and

$$\bar{x} = x - y_1 \left(\frac{1 + y_1^2}{y_2} \right) \text{ and}$$

$$\bar{y} = y + \left(\frac{1 + y_1^2}{y_2} \right)$$

1. Find the centre of ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ at (a, b) .

Soln:

Given curve: $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ -----> (1)

Point: $P(a, b)$

Differentiating (1) w.r.t 'x', we get

$$\frac{1}{a^2} 2x + \frac{1}{b^2} 2y \frac{dy}{dx} = 0$$
$$\Rightarrow \frac{dy}{dx} = \frac{-x}{y} \cdot \frac{b^2}{a^2} \text{ -----> (2)}$$

$$\therefore \frac{dy}{dx} \text{ at } (a, b) = \frac{-a}{b} \cdot \frac{b^2}{a^2} = \frac{-b}{a}$$

$$y = \frac{-b}{a}$$

From (2),

$$\frac{d^2 y}{dx^2} = \frac{d}{dx} \left[\frac{dy}{dx} \right]$$



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$$= \frac{d}{dx} \left[\frac{-x}{y} \cdot \frac{b^2}{a^2} \right]$$

$$= \frac{-b^2}{a^2} \frac{d}{dx} \left[\frac{x}{y} \right]$$

$$\text{i.e., } \frac{d^2 y}{dx^2} = -\frac{b^2}{a^2} \left[\frac{y(1) - x \frac{dy}{dx}}{y^2} \right]$$

$$\therefore \frac{d^2 y}{dx^2} \text{ at } P(a, b) = \frac{-b^2}{a^2} \left[\frac{b - a \left(\frac{-b}{a} \right)}{b^2} \right]$$

$$y_2 = \frac{2b}{a^2}$$

$$\text{Consider } \frac{1 + y_1^2}{y_2} = \frac{1 + \left(\frac{-b}{a} \right)^2}{\frac{-2b}{a^2}} = \frac{a^2 + b^2}{-2b} = \frac{-(a^2 + b^2)}{+2b}$$

$$\bar{x} = a - \left(\frac{-b}{a} \right) \left(\frac{a^2 + b^2}{2b} \right) = \frac{2a^2 - a^2 - b^2}{2a} = \frac{a^2 - b^2}{2a}$$

$$\bar{y} = b - \left(\frac{a^2 + b^2}{2b} \right) = \frac{2b^2 - a^2 - b^2}{2b} = -\left(\frac{a^2 - b^2}{2b} \right)$$

$$\therefore \text{Centre of curvature } C(\bar{x}, \bar{y}) \text{ at } P(a, b) \text{ is } \bar{x} = \frac{a^2 + b^2}{2a}, \bar{y} = \frac{b^2 - a^2}{2b}.$$



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2. Find the centre of curvature of hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ at the point $(a \sec \theta, b \tan \theta)$.

Soln:

$$\text{Given curve: } \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \quad \text{-----> (1)}$$

$$\text{Point: } P(a \sec \theta, b \tan \theta)$$

Here given point is the parametric representation of (1).

$$\therefore x = a \sec \theta, \quad \text{-----> (2)}$$

$$y = b \tan \theta, \quad \text{-----> (3) where } \theta \text{ is the parameter.}$$

$$\frac{dx}{d\theta} = a \sec \theta \tan \theta, \quad \frac{dy}{d\theta} = b \sec^2 \theta$$

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{b \sec^2 \theta}{a \sec \theta \tan \theta} = \frac{b}{a} \cdot \frac{1}{\cos \theta} \cdot \frac{\cos \theta}{\sin \theta}$$

$$y_1 = \frac{b}{a} \operatorname{cosec} \theta$$

$$\frac{d^2 y}{dx^2} = \frac{d}{d\theta} \left(\frac{dy}{dx} \right) \cdot \frac{d\theta}{dx}$$

$$= \frac{d}{d\theta} \left(\frac{b}{a} \operatorname{cosec} \theta \right) \cdot \frac{1}{\frac{dx}{d\theta}}$$

$$= \frac{b}{a} (-\operatorname{cosec} \theta \cot \theta) \cdot \frac{1}{a \sec \theta \tan \theta}$$



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$$= \frac{-b}{a^2} \left[\frac{1}{\sin \theta} \cdot \frac{\cos \theta}{\sin \theta} \cdot \cos \theta \cdot \frac{\cos \theta}{\sin \theta} \right]$$

$$= \frac{-b \cos^3 \theta}{a^2 \sin^3 \theta}$$

$$y_2 = \frac{-b}{a^2} \cot^3 \theta$$

Consider

$$\left(\frac{1+y_1^2}{y_2} \right) = \frac{1 + \left(\frac{b}{a} \right)^2 \operatorname{cosec}^2 \theta}{\frac{-b}{a^2} \cot^3 \theta} = \frac{-(a^2 + b^2 \operatorname{cosec}^2 \theta)}{b \cot^3 \theta}$$

$$\therefore \bar{x} = a \sec \theta + \left(\frac{b}{a} \right) \operatorname{cosec} \theta \left(\frac{a^2 + b^2 \operatorname{cosec}^2 \theta}{b \cot^3 \theta} \right)$$

$$= a \sec \theta + \frac{b}{a} \cdot \frac{1}{\sin \theta} \left(\frac{a^2 + b^2 \frac{1}{\sin^2 \theta}}{b \cdot \frac{\cos^3 \theta}{\sin^3 \theta}} \right)$$

$$= a \sec \theta + \frac{1}{a} \left(\frac{a^2 \sin^2 \theta + b^2}{\cos^3 \theta} \right)$$

$$= \frac{a^2 \cos^2 \theta + a^2 \sin^2 \theta + b^2}{a \cos^3 \theta}$$

$$\bar{x} = \left(\frac{a^2 + b^2}{a} \right) \sec^3 \theta$$



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Similarly,

$$\begin{aligned}\bar{y} &= b \tan \theta - \left[\frac{a^2 + b^2 \operatorname{cosec}^2 \theta}{b \cot^3 \theta} \right] \\ &= b \tan \theta - \left[\frac{a^2 + b^2 \frac{1}{\sin^2 \theta}}{b \frac{\cos^3 \theta}{\sin^3 \theta}} \right] \\ &= b \frac{\sin \theta}{\cos \theta} - \left[\frac{a^2 \sin^2 \theta + b^2}{b \frac{\cos^3 \theta}{\sin \theta}} \right] \\ &= \frac{b \sin \theta}{\cos \theta} - \frac{\sin \theta}{b \cos^3 \theta} [a^2 \sin^2 \theta + b^2] \\ &= \frac{b^2 \sin \theta \cos^2 \theta - \sin \theta (a^2 \sin^2 \theta + b^2)}{b \cos^3 \theta} \\ &= \frac{b^2 \sin \theta (1 - \sin^2 \theta) - a^2 \sin^3 \theta - b^2 \sin \theta}{b \cos^3 \theta} \\ &= \frac{b^2 \sin \theta - b^2 \sin^3 \theta - a^2 \sin^3 \theta - b^2 \sin \theta}{b \cos^3 \theta} \\ &= \frac{-(a^2 + b^2) \sin^3 \theta}{b \cos^3 \theta} \\ \therefore \bar{y} &= - \left(\frac{a^2 + b^2}{b} \right) \tan^3 \theta\end{aligned}$$