Unit - II

Fourier Transform

Fourier transform pair

Fourier transform of $f(x)$ is defined as,

$$F[f(x)] = F(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{isx}\, dx \longrightarrow (1)$$

Inverse fourier transform

Inverse Fourier transform is defined as,

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F[S]\, e^{-isx}\, ds \longrightarrow (2)$$

Equation (1) & (2) Together known as fourier Transform pair

Parseval Identity

$$\int_{-\infty}^{\infty} |F(s)|^2\, ds = \int_{-\infty}^{\infty} |f(x)|^2\, dx$$

Note

1. $e^{i\theta} = \cos\theta + i\sin\theta$
2. $e^{-i\theta} = \cos\theta - i\sin\theta$
3. $\sin^2\theta = \dfrac{1-\cos 2\theta}{2}$

1. Find the Fourier transform of $f(x) = \begin{cases} a-|x|, & |x|<a \\ 0 & |x|>a \end{cases}$
Hence deduce that i) $\int_0^{\infty} \left[\frac{\sin t}{t}\right]^2 dt = \frac{\pi}{2}$
ii) $\int_0^{\infty} \left[\frac{\sin t}{t}\right]^4 dt = \frac{\pi}{3}$

Sol

$$F[f(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{isx}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} [a-|x|]\, e^{isx}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} [a-|x|]\,(\cos sx + i\sin sx)\, dx$$

$$= \frac{1}{\sqrt{2\pi}}\left[\int_{-a}^{a}[a-|x|]\cos sx\,dx + \int_{-a}^{a}[a-|x|]\, i\sin sx\, dx\right]$$

$$= \frac{2}{\sqrt{2\pi}}\int_0^a (a-x)\cos sx$$

$u = a-x \qquad v = \cos sx$

$u' = 0-1 \qquad v_1 = \frac{\sin sx}{s}$

$u'' = 0 \qquad v_2 = -\frac{\cos sx}{s^2}$

$$F[f(x)] = \frac{2}{\sqrt{2\pi}}\left[(a-x)\frac{\sin sx}{s} - \frac{\cos sx}{s^2}\right]_0^a$$

$$= -\frac{2}{\sqrt{2\pi}}\left[\frac{\cos sx}{s^2}\right]_0^a$$

$$= -\frac{2}{\sqrt{2\pi}}\left(\frac{\cos sa}{s^2} - \frac{1}{s^2}\right)$$

$$= \frac{2}{\sqrt{2\pi}}\left(\frac{1-\cos sa}{s^2}\right)$$

Using Inverse fourier transform

$$f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} F(s)\, e^{-isx}\, ds$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{2}{\sqrt{2\pi}}\left(\frac{1-\cos sa}{s^2}\right)e^{isx}\, ds$$

$$= \frac{2}{2\pi}\int_{-\infty}^{\infty}\frac{1-\cos sa}{s^2}(\cos sx - i\sin sx)\, ds$$

$$= \frac{1}{\pi}\int \frac{1-\cos sa}{s^2}\cos sx$$

$$- \int \frac{1-\cos sa}{s^2}\, i\sin sx$$

$$a-|x| = \frac{2}{\pi}\int_0^{\infty}\left(\frac{1-\cos sa}{s^2}\right)\cos sx\, ds$$

Put $a=2$, $s=t$, $x=0$

$$2-0 = \frac{2}{\pi}\int_0^{\alpha} \frac{1-\cos 2t}{t^2}\, dt$$

$$\cancel{2} = \frac{\cancel{2}}{\pi}\int_0^{\alpha} \frac{2\sin^2 t}{t^2}\, dt \qquad \sin^2\theta = \frac{1-\cos 2\theta}{2}$$

$$1 = \frac{2}{\pi}\int_0^{\alpha} \frac{\sin^2 t}{t^2}\, dt$$

$$\int_0^{\alpha} \left(\frac{\sin t}{t}\right)^2 dt = \frac{\pi}{2}$$

ii) Using Parseval's identity

$$\int_{-\alpha}^{\alpha} [F(s)]^2 ds = \int_{-\alpha}^{\alpha} [f(x)]^2 dx$$

$$\int_{-\alpha}^{\alpha} \left[\sqrt{\frac{2}{\pi}}\, \frac{1-\cos sa}{s^2}\right]^2 ds = \int_{-a}^{a} [a-|x|]^2 dx$$

$$\frac{2}{\pi}\, 2\int_0^{\alpha} \left[\frac{1-\cos sa}{s^2}\right]^2 ds = 2\int_0^{a} (a-x)^2 dx$$

Put $a=2,\ s=t$

$$\frac{4}{\pi}\int_0^{\alpha} \left[\frac{1-\cos 2t}{t^2}\right]^2 dt = 2\int_0^{2} (2-x)^2 dx$$

$$\frac{4}{\pi}\int_0^{\alpha} \left(\frac{2\sin^2 t}{t^2}\right)^2 dt = 2\left[\frac{(2-x)^3}{-3}\right]_0^2$$

$$\frac{16}{\pi}\int_0^{\alpha} \frac{\sin^4 t}{t^4}\, dt = -\frac{2}{3}(0-8)$$

$$\frac{\cancel{16}}{\pi}\int_0^{\alpha} \left[\frac{\sin t}{t}\right]^4 dt = \frac{\cancel{16}}{3}$$

$$\int_0^{\alpha} \left(\frac{\sin t}{t}\right)^4 dt = \frac{\pi}{3}$$