2. Find the Fourier transform for

$$f(x) = \begin{cases} a^2 - x^2 & |x| < a \\ 0 & |x| > a > 0 \end{cases}$$ Hence deduce that

i) $\displaystyle\int_0^\infty \frac{\sin t - t\cos t}{t^3}\,dt = \frac{\pi}{4}$

ii) $\displaystyle\int_0^\infty \left[\frac{\sin t - t\cos t}{t^3}\right]^2 dt = \frac{\pi}{15}$

Sol

$$F[f(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{isx}\,dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} (a^2 - x^2)\, e^{isx}\,dx$$

$$= \frac{1}{\sqrt{2\pi}} \left\{ \int_{-a}^{a} (a^2 - x^2)(\cos x + i\sin sx)\,dx \right.$$

$$= \frac{2}{\sqrt{2\pi}} \int_0^a (a^2 - x^2) \cos sx \,dx$$

$u = a^2 - x^2$ $\quad\quad$ $v = \cos sx$

$u' = -2x$ $\quad\quad$ $v_1 = \dfrac{\sin sx}{s}$

$u'' = -2$ $\quad\quad$ $v_2 = \dfrac{-\cos sx}{s^2}$

$v_3 = \dfrac{-\sin sx}{s^3}$

$$= \frac{2}{\sqrt{2\pi}} \left[(a^2 - x^2) \frac{\sin sx}{s} - \frac{2x \cos sx}{s^2} + \frac{2\sin sx}{s^3} \right]_0^a$$

$$= \frac{2}{\sqrt{2\pi}} \left[-\frac{2a\cos sa}{s^2} + \frac{2\sin sa}{s^3} \right]$$

$$= \frac{4}{\sqrt{2\pi}} \left[\frac{-as\cos sa + \sin sa}{s^3} \right]$$

Using inverse Fourier Transform

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\alpha} F(s)\, e^{-isx}\, ds$$

$$a^2 - x^2 = \frac{1}{\sqrt{2\pi}} \int_{-\alpha}^{\alpha} \frac{4}{\sqrt{2\pi}} \left[\frac{-as\cos sa + \sin sa}{s^3} \right]$$

$$(\cos sx - i \sin sx)\, ds$$

$$= \frac{4}{2\pi} \left[\int_{-\alpha}^{\alpha} \frac{-as\cos sa + \sin sa}{s^3} \right] \cos sx \; ds$$

$$= \frac{2}{\pi} \times 2 \int_0^{\alpha} \frac{-as\cos sa + \sin sa}{s^3} \cos s(x)\, ds$$

put $s = t$, $x = 0$, $a = 1$

$$1 - 0 = \frac{4}{\pi} \int_0^{\alpha} \left[\frac{-t\cos t + \sin t}{t^3} \right] dt$$

$$\frac{\pi}{4} = \int_0^{\alpha} \sin t -$$

$$\int_0^{\alpha} \left[\frac{\sin t - t\cos t}{t^3} \right] dt = \frac{\pi}{4}$$

ii) Using parseval's identity

$$\int_{-\alpha}^{\alpha} [F(s)]^2\, ds = \int_{-\alpha}^{\alpha} [f(x)]^2\, dx$$

$$\int_{-\alpha}^{\alpha} \left[\frac{4}{\sqrt{2\pi}} \left[\frac{\sin sa - as\cos sa}{s^3} \right] \right]^2 ds = \int_{-a}^{a} (a^2 - x^2)^2\, dx$$

Put $s = t$, $a = 1$

$$\frac{16}{2\pi} \times 2 \int_0^{\alpha} \left[\frac{\sin t - t\cos t}{t^3} \right]^2 dt = 2 \int_0^1 (1 - x^2)^2\, dx$$

$$= 2 \int_0^1 (1 - 2x^2 + x^4)\, dx$$