5]. $F[x^n f(x)] = (-i)^n \frac{d^n}{ds^n} F(s)$

Proof :

$$F[S] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{isx}\, dx$$

Differentiating both sides 'n' times with respect to 's'

$$\frac{d^n}{ds^n} F(s) = \frac{1}{\sqrt{2\pi}} \frac{d^n}{ds^n} \int_{-\infty}^{\infty} f(x)\, e^{isx}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{\partial^n}{\partial s^n} [f(x) e^{isx}]\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, (ix)^n\, e^{isx}\, dx$$

$$= i^n \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, x^n\, e^{isx}\, dx$$

$$= i^n F[x^n f(x)]$$

$$\Rightarrow F[x^n f(x)] = \frac{1}{(i)^n} \frac{d^n}{ds^n} F(s)$$

$$= (-i)^n \frac{d^n}{ds^n} F(s)$$

6]. i). $F[f'(x)] = -is\, F(s)$ if $f(x) \to 0$ as $x \to \pm\infty$

ii). $F[f^{(n)}(x)] = (-i)^n s^n F(s)$ if $f(x), f'(x), \ldots$
$f^{(n-1)}(x) \to 0$ as $x \to \pm\infty$

7]. $F[\overline{f(x)}] = \overline{F(-s)}$

8]. i) $F_S[x f(x)] = -\frac{d}{ds} F_C[f(x)]$

ii) $F_C[x f(x)] = \frac{d}{ds} F_S[f(x)]$

Problems on properties.

1]. Find the F.S. and F.C.T. of $x e^{-ax}$.

Soln.:

By property,

$$F_S[x f(x)] = -\frac{d}{ds} F_C[f(x)]$$

$$F_S[x e^{-ax}] = -\frac{d}{ds} F_C[e^{-ax}]$$

$$= -\frac{d}{ds} \sqrt{\frac{2}{\pi}} \frac{a}{a^2+s^2}$$

$$F_S[x e^{-ax}] = \frac{\sqrt{2}}{\pi} + \frac{a2s}{(a^2+s^2)^2}$$

and

$$F_C[x f(x)] = \frac{d}{ds} F_S[f(x)]$$

$$F_C[x e^{-ax}] = \frac{d}{ds} F_S[e^{-ax}]$$

$$= \frac{d}{ds}\left[\sqrt{\frac{2}{\pi}} \frac{s}{s^2+a^2}\right]$$

$$= \sqrt{\frac{2}{\pi}}\left[\frac{(s^2+a^2)(1) - s(2s)}{(s^2+a^2)^2}\right]$$

$$= \sqrt{\frac{2}{\pi}}\left[\frac{s^2+a^2-2s^2}{(s^2+a^2)^2}\right]$$

$$= \sqrt{\frac{2}{\pi}}\left[\frac{a^2-s^2}{(s^2+a^2)^2}\right]$$

H.w. Find the FCT of $e^{-a^2x^2}$ and hence find $F_C[x e^{-a^2x^2}]$.