Parseval's Identity:

1. Using transform methods, evaluate $\int_0^\infty \frac{dx}{(x^2+a^2)^2}$

Soln.:

consider $f(x) = e^{-ax}$

$$F_c[f(x)] = F_c[e^{-ax}] = \sqrt{\frac{2}{\pi}}\,\frac{a}{s^2+a^2}$$

Now, $\int_0^\infty |f(x)|^2\,dx = \int_0^\infty |F_c(s)|^2\,ds$

$$\int_0^\infty e^{-2ax}\,dx = \int_0^\infty \frac{2}{\pi}\,\frac{a^2}{(s^2+a^2)^2}\,ds$$

$$\left[\frac{e^{-2ax}}{-2a}\right]_0^\infty = \frac{2a^2}{\pi}\int_0^\infty \frac{ds}{(s^2+a^2)^2}$$

$$\frac{-1}{2a}[0-1] = \frac{2a^2}{\pi}\int_0^\infty \frac{ds}{(s^2+a^2)^2}$$

$$\frac{\pi}{2a(2a^2)} = \int_0^\infty \frac{ds}{(s^2+a^2)^2}$$

$$\Rightarrow \int_0^\infty \frac{dx}{(x^2+a^2)^2} = \frac{\pi}{4a^3}$$

2]. Using transform methods, evaluate $\int_0^\infty \frac{x^2\,dx}{(x^2+a^2)^2}$ where $a>0$.

Soln. :

consider $f(x) = e^{-ax}$

$$F_s[f(x)] = F_s[e^{-ax}] = \sqrt{\frac{2}{\pi}}\frac{s}{s^2+a^2}$$

Now,

$$\int_0^\infty |f(x)|^2\,dx = \int_0^\infty |F_s(s)|^2\,ds$$

$$\int_0^\infty e^{-2ax}\,dx = \int_0^\infty \frac{2}{\pi}\frac{s^2}{(s^2+a^2)^2}\,ds$$

$$\left[\frac{e^{-2ax}}{-2a}\right]_0^\infty = \frac{2}{\pi}\int_0^\infty \frac{s^2}{(s^2+a^2)^2}\,ds$$

$$\frac{-1}{2a}[0-1] = \frac{2}{\pi}\int_0^\infty \frac{s^2}{(s^2+a^2)^2}\,ds$$

$$\Rightarrow \int_0^\infty \frac{s^2}{(s^2+a^2)^2}\,ds = \frac{\pi}{4a} \Rightarrow \int_0^\infty \frac{x^2}{(x^2+a^2)^2}\,dx = \frac{\pi}{4a}$$

Find the F.T. of $e^{-a|x|}$ using PI

Show that $\int_0^\infty \frac{dx}{(x^2+1)^2} = \frac{\pi}{4}$

Soln.:

$$F[e^{-|x|}] = \sqrt{\frac{2}{\pi}} \frac{1}{1+s^2}$$ (Problem 2 Already Proved)

By Parseval's Identity,

$$\int_{-\infty}^{\infty} |F(s)|^2 ds = \int_{-\infty}^{\infty} |f(x)|^2 dx$$

$$\int_{-\infty}^{\infty} \frac{2}{\pi} \frac{1}{(1+s^2)^2} ds = \int_{-\infty}^{\infty} (e^{-|x|})^2 dx$$

$$\frac{4}{\pi}\int_0^\infty \frac{ds}{(s^2+1)^2} = 2\int_0^\infty e^{-2x} dx$$

$$= 2\left(\frac{e^{-2x}}{-2}\right)_0^\infty$$

$$= -1(0-1)$$

$$= 1$$

$$\Rightarrow \int_0^\infty \frac{ds}{(s^2+1)^2} = \frac{\pi}{4}$$

$$\Rightarrow \int_0^\infty \frac{dx}{(x^2+1)^2} = \frac{\pi}{4}$$