ONE DIMENSIONAL WAVE EQUATION:

General form: $\frac{\partial^2 y}{\partial t^2} = a^2 \frac{\partial^2 y}{\partial x^2}$ where $a^2 = \frac{\text{Tension}}{\text{mass per unit length of the string}}$

* In one dimensional wave equation $y = y(x,t)$ is displacement of a particle in the string.
* Velocity, $\frac{\partial y}{\partial t} = \frac{\partial y}{\partial t}(x,t)$
* $y(x,0)$ is called initial displacement of a particle

Possible solution of one dimensional eqn. are:

$y(x,t) = (Ae^{ax} + Be^{-ax})(Ce^{pat} + De^{-pat})$

$y(x,t) = (A\cos px + B\sin px)(C\cos pat + D\sin pat)$

$y(x,t) = (Ax+B)(Ct+D)$

Suitable solution of one dimensional wave eqn.:

$y(x,t) = (A\cos px + B\sin px)(C\cos pat + D\sin pat)$

TYPE I: Vibrating string with zero initial velocity:

(i) $y(0,t) = 0$ (iii) $\frac{\partial y}{\partial t}(x,0) = 0$

(ii) $y(l,t) = 0$ (iv) $y(x,0) = f(x)$

(1) A string is streched and fastened to extreme pts. $x=0$ and $x=l$. Motion is started by displacing the string into the form $y = K(lx - x^2)$ from which it is released at time $t=0$. Find the displacement at any pt on the string at a distance of 'x' from one end at a time 't'.

Soln: The general form of one dimensional wave eqn. is

$$\frac{\partial^2 y}{\partial t^2} = a^2 \frac{\partial^2 y}{\partial x^2} \quad \text{---} (1)$$

Suitable soln. for one dimensional wave eqn is

$$y(x,t) = (A \cos px + B \sin px)(C \cos pat + D \sin pat) \quad \text{---} (2)$$

Boundary conditions :

(i) $y(0,t) = 0$

(ii) $y(l,t) = 0$

(iii) $\frac{\partial y}{\partial t}(x,0) = 0$ &

(iv) $y(x,0) = k(lx - x^2)$

By cdtn. (i) in (2) we have,

$$y(0,t) = (A \cos 0 + B \sin 0)(C \cos pat + D \sin pat)$$

$$0 = A(C \cos pat + D. \sin pat)$$

$$\Rightarrow \boxed{A = 0}$$

Sub $A=0$ in (2).

$y(x,t) = B\sin px\,(C\cos pat + D\sin pat) \rightarrow (3)$

By cdtn. (ii) in (3) we have,

$y(l,t) = B\sin pl\,(C\cos pat + D\sin pat)$

$0 = B\sin pl\,.(C\cos pat + D\sin pat)$

$B\sin pl = 0$

If $B=0$; suitable soln. is zero

If $B \neq 0$ and ~~...~~; $\sin pl = 0$

$pl = n\pi$

$\boxed{p = \frac{n\pi}{l}}$

Sub $p = \frac{n\pi}{l}$ in (3)

$y(x,t) = B\sin\frac{n\pi}{l}x\left[C\cos\frac{n\pi}{l}at + D\sin\frac{n\pi}{l}at\right] \rightarrow (4)$

By cdtn. (iii) in (4),

w.r.t t } $\frac{\partial y}{\partial t}(x,t) = B\sin\frac{n\pi x}{l}\left[-C\sin\frac{n\pi}{l}at\cdot\left(\frac{n\pi a}{l}\right) + D\cos\frac{n\pi}{l}at\cdot\left(\frac{n\pi a}{l}\right)\right]$

$$= B\sin\frac{n\pi x}{l}\left(\frac{n\pi a}{l}\right)\left[-C\sin\frac{n\pi a}{l}t + D\cos\frac{n\pi a}{l}t\right]$$

$$\frac{\partial y}{\partial t}(x,0) = B\sin\frac{n\pi x}{l}\cdot\frac{n\pi a}{l}\left[-C\sin\frac{n\pi a}{l}(0) + D\cos\frac{n\pi a}{l}(0)\right]$$

$$= B\sin\frac{n\pi x}{l}\cdot\frac{n\pi a}{l}[D]$$

$$\frac{\partial y}{\partial t}(x,0) = BD\sin\frac{n\pi x}{l}\cdot\frac{n\pi a}{l}$$

$$0 = BD\sin\frac{n\pi x}{l}\cdot\frac{n\pi a}{l} \Rightarrow D = 0$$

Sub $D=0$ in (4)

$$y(x,t) = \left(B\sin\frac{n\pi x}{l}\right)\left(C\cos\frac{n\pi}{l}at\right) = BC\sin\frac{n\pi x}{l}\cos\frac{n\pi}{l}at$$

General soln. is

$$y(x,t) = \sum_{n=1}^{\infty} BC\sin\frac{n\pi x}{l}\cos\frac{n\pi}{l}at$$

$$y(x,t) = \sum_{n=1}^{\infty} B_n\sin\frac{n\pi x}{l}\cos\frac{n\pi a}{l}t \quad \text{—— (5)} \quad \boxed{BC = B_n}$$

By cdtn. (iv) in (5)

$$y(x,0) = \sum_{n=1}^{\infty} B_n\sin\frac{n\pi x}{l}$$

$$y = k(lx - x^2) = \sum_{n=1}^{\infty} B_n\sin\frac{n\pi}{l}x \quad \text{—— (6)}$$

$$k(lx - x^2) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l} \quad \text{--- (7)}$$

By Comparing (6) & (7), $B_n = b_n$.

$$\therefore B_n = \frac{2}{l} \int_0^l f(x) \sin \frac{n\pi x}{l}\, dx.$$

$$= \frac{2}{l} \int_0^l k(lx - x^2) \sin \frac{n\pi x}{l}\, dx$$

$$= \frac{4kl^2}{n^3\pi^3} \left[1 - (-1)^n\right]$$

$$B_n = \begin{cases} 0, & n \text{ is even} \\ \dfrac{8kl^2}{n^3\pi^3}, & n \text{ is odd}. \end{cases}$$

Sub. B_n in (5),

$$y(x,t) = \sum_{n=\text{odd}}^{\infty} \frac{8kl^2}{n^3\pi^3} \sin \frac{n\pi x}{l} \cos \frac{n\pi a}{l} t.$$