TYPE - II . VIBRATING STRING WITH NON-ZERO INITIAL VELOCITY.

The initial and boundary cdtns. of $y(x,t)$ are:

(i) $y(0,t)=0$

(ii) $y(l,t)=0$

(iii) $y(x,0)=0$

(iv) $\frac{\partial y}{\partial t}(x,0)=f(x)$.

If a string of length 'l' is initially at rest in its equilibrium position and each of its pts is gn. a velocity 'v' such that.

$$v=\begin{cases} cx, & 0<x<l/2 \\ c(l-x), & l/2<x<l \end{cases}$$

Find the displacement of $y(x,t)$ at any time 't'.

Soln: The boundary cdtns. are

(i) $y(0,t)=0, \quad t\geq 0$

(ii) $y(l,t)=0$

(iii) $y(x,0)=0$

(iv) $\frac{\partial y}{\partial t}(x,0)=\begin{cases} cx, & 0\leq x\leq l/2 \\ c(l-x), & l/2\leq x\leq l. \end{cases}$

The suitable eqn. is

$$y(x,t)=(A\cos px+B\sin px)(C\cos pat+D\sin pat) \quad \text{--- (1)}$$

Apply (i) in ①,

$y(0,t) = A\,(C\cos pat + D\sin pat)$

$0 = A\,(C\cos pat + D\sin pat)$

$\Rightarrow A = 0$ sub $A=0$ in ①, $y(x,t) = B\sin px\,[C\cos pat + D\sin pat]$ — ②

Apply (ii) in ②,

$y(l,t) = B\sin pl\,[C\cos pat + D\sin pat]$

$0 = B\sin pl\,[C\cos pat + D\sin pat]$

$B \neq 0 \quad \sin pl = 0$

$p = \frac{n\pi}{l}$

Sub in ②. $y(x,t) = B\sin\frac{n\pi}{l}x\left[C\cos\frac{n\pi}{l}at + D\sin\frac{n\pi}{l}at\right]$ — ③

Apply (iii) in ③,

$y(x,0) = B\sin\frac{n\pi}{l}x\,[C]$

$0 = BC\sin\frac{n\pi}{l}x$

$B \neq 0,\ C = 0$

Sub $C=0$ in ③

$y(x,t) = \left(B\sin\frac{n\pi}{l}x\right)\left(D\sin\frac{n\pi}{l}at\right)$

$y(x,t) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi}{l}x\,\sin\frac{n\pi}{l}at$ — ④

Apply (iv) in (4)

$$\frac{\partial y}{\partial t}(x,t) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi}{l}x \cos\frac{n\pi}{l}at\left(\frac{n\pi a}{l}\right)$$

$$\frac{\partial y}{\partial t}(x,0) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi}{l}x\left(\frac{n\pi}{l}a\right)$$

$$\begin{cases} cx, & 0<x<l/2 \\ c(l-x), & l/2<x<l \end{cases} = \frac{n\pi a}{l}\sum_{n=1}^{\infty} B_n \sin\frac{n\pi}{l}x = \sum_{n=1}^{\infty} C_n \sin\frac{n\pi}{l}x$$

where $C_n = \left(\frac{n\pi}{l}a\right)B_n$

$$B_n = \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi}{l}x\,dx.$$

$$= \frac{2}{l}\left[\int_0^{l/2} cx\sin\frac{n\pi}{l}x\,dx + \int_{l/2}^{l} c(l-x)\sin\frac{n\pi}{l}x\,dx\right] = \frac{4cl}{n^2\pi^2}\sin n\pi/2$$

$$B_n = \frac{l}{n\pi a}\cdot\frac{4cl}{n^2\pi^2}\sin n\pi/2$$

$$B_n = \frac{2c}{n\pi a}\left(\frac{2\sin n\pi/2}{(n\pi/l)^2}\right) = \frac{4l^2c}{n^3\pi^3 a}(\sin n\pi/2)$$

b B_n in (4).

$$y(x,t) = \sum_{n=1}^{\infty}\frac{4l^2c}{n^3\pi^3 a}\sin n\pi/2 \sin\frac{n\pi}{l}x \sin\frac{n\pi}{l}at$$

HW: [Vibrating string with non zero velocity]

If a string of length l is initially at rest in equilibrium position & each pt. of it is gn. the velocity $\left(\frac{\partial y}{\partial t}\right)_{t=0} = v_0 \sin^3 \frac{\pi x}{l}$, $0<x<l$. Determine the transverse displacement $y(x,t)$.

Soln: By Applying (i), (ii) & (iii) boundary cdtns. we get

$$y(x,t) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{l} \sin\frac{n\pi a t}{l}$$

Apply (iv) we get,

$$\frac{\partial y}{\partial t}(x,t) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{l}\left(\cos\frac{n\pi a t}{l}\right)\cdot\frac{n\pi a}{l}$$

$$\left(\frac{\partial y}{\partial t}\right)(x,0) = \sum_{n=1}^{\infty} B_n \frac{n\pi a}{l} \sin\frac{n\pi x}{l}$$

$$v_0 \sin^3\frac{\pi x}{l} = \sum_{n=1}^{\infty} B_n \frac{n\pi a}{l} \sin\frac{n\pi x}{l}$$

$$\frac{v_0}{4}\left[3\sin\frac{\pi x}{l} - \sin\frac{3\pi x}{l}\right] = B_1 \frac{\pi a}{l}\sin\frac{\pi x}{l} + B_2\frac{2\pi a}{l}\sin\frac{2\pi x}{l} + B_3 \sin\frac{3\pi x}{l}\,\frac{3\pi a}{l} + \cdots$$

Equating like terms OBS we get

$$B_1 \cdot \frac{\pi a}{l} = \frac{3v_0}{4} \Rightarrow B_1 = \frac{3lv_0}{4\pi a}.$$

$$B_2 = 0.$$

$$B_3 \cdot \frac{3\pi a}{l} = -\frac{v_0}{4} \Rightarrow B_3 = -\frac{v_0 l}{12\pi a}.$$

$$B_4 = 0$$

(ie) $B_n = 0$ for $n \neq 1, 3$.

$$\therefore y(x,t) = \frac{3lv_0}{4\pi a}\sin\frac{\pi x}{l}\sin\frac{\pi a t}{l} - \frac{v_0 l}{12\pi a}\sin\frac{3\pi x}{l}\sin\frac{3\pi a t}{l}.$$