One Dimensional Heat Equation :

The general form of one-dimensional heat equation is $\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2}$ where $\alpha^2 = \frac{k}{s\rho}$

k - Conductivity
s - Specific heat
ρ - Density.

Possible soln. of one-dimensional heat eqn:

(i) $u(x,t) = A(Bx+C) = c_1(c_2x+c_3)$

(ii) $u(x,t) = Ae^{\alpha^2p^2t}(Be^{px}+Ce^{-px}) = c_1e^{\alpha^2p^2t}(c_2e^{px}+c_3e^{-px})$

(iii) $u(x,t) = Ae^{-\alpha^2p^2t}(B\cos px + C\sin px)$.
$= c_1e^{-\alpha^2p^2t}(c_2\cos px + c_3\sin px)$

Suitable soln. is

$$u(x,t) = (A\cos px + B\sin px)e^{-\alpha^2p^2t} \quad \boxed{c_1c_2 = A,\ c_1c_3 = B}$$

The initial and boundary condtns. are

(i) $u(0,t) = 0$

(ii) $u(l,t) = 0$

(iii) $u(x,0) = f(x)$.

STEADY STATE CONDITIONS AND ZERO BOUNDARY CONDITIONS

Defn: The state in which the temp. at any point in the body does not vary with respect to time t is called steady state.

$\therefore$ $u(x,t)$ becomes $u(x)$ under the steady state cdtn

$$\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2} \quad \text{—} ①$$

In steady state $\frac{\partial u}{\partial t} = 0$

$\therefore$ ① becomes $\alpha^2 \frac{\partial^2 u}{\partial x^2} = 0$

$$\Rightarrow \frac{\partial^2 u}{\partial x^2} = 0 \quad (\alpha \neq 0).$$

$\therefore$ General soln. is $u(x) = ax + b$.

(Q1)

1) A rod of length l has its ends A and B kept at 0°c and 100°c until steady state cdtn prevail. If the temp. at B is reduced suddenly to 0°c and kept so while that of A is maintained, find the temp $u(x,t)$ at a distance x from A and at time t.

Soln: The one dimensional heat eqn is

$$\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2}$$

when steady state cdtns prevails $\frac{\partial u}{\partial t} = 0$

We get $\frac{\partial^2 u}{\partial x^2} = 0$

$$u(x) = ax + b$$

$$\left[\begin{array}{ll} \text{when at } x=0 & ; u(0) = 0 \\ \text{when } x = l & ; u(l) = 100 \end{array}\right\}$$

The boundary cdtns are

(i) $u(0) = 0$

(ii) $u(l) = 100$

To find $u(x)$ we need the following:

$$u(x) = ax + b$$

$$u(0) = a(0) + b$$

$$\boxed{0 = b}$$

$$\therefore u(x) = ax$$

$$u(l) = al$$

$$100 = al \Rightarrow \boxed{a = \frac{100}{l}}$$

$\therefore$ (iii) $u(x) = \frac{100x}{l}$

Now the end B is reduced to zero.

At this stage steady state $\rightarrow$ unsteady state

$\therefore$ Initial temp. dist for unsteady state is

$$u(x) = \frac{100x}{l}$$

The heat eqn. is $\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2}$

The boundary cdtns are:

(a) $u(0,t) = 0$ for all $t > 0$

(b) $u(l,t) = 0$ for all $t > 0$

(c) $u(x,0) = \frac{100x}{l}$ for x in $(0,l)$

The suitable soln. is

$$u(x,t) = (A\cos px + B\sin px)\, e^{-\alpha^2 p^2 t} \quad \text{—①}$$

Apply (a) in ① we get

$$u(0,t) = A e^{-\alpha^2 p^2 t} \Rightarrow$$

$$0 = A e^{-\alpha^2 p^2 t}$$

$$\Rightarrow \boxed{A = 0}$$

$$① \Rightarrow u(x,t) = B \sin px \, e^{-\alpha^2 p^2 t}$$

Applying (b) in ① we get

$$u(l,t) = B \sin pl \, e^{-\alpha^2 p^2 t}$$

$$\Rightarrow 0 = B \sin pl \, e^{-\alpha^2 p^2 t}$$

$$\Rightarrow B \neq 0, \quad p = \frac{n\pi}{l}$$

$$① \Rightarrow u(x,t) = B \sin \frac{n\pi}{l} x \, e^{-\frac{\alpha^2 n^2 \pi^2}{l^2} t}$$

∴ The general soln. is

$$u(x,t) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi}{l} x \, e^{-\frac{\alpha^2 n^2 \pi^2 t}{l^2}}$$

Applying (c) in the above.

$$u(x,0) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{l}$$

$$\frac{100x}{l} = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi}{l} x$$

$$B_n = \frac{2}{l} \int_0^l \frac{100x}{l} \sin \frac{n\pi}{l} x \, dx$$

$$= \frac{200}{l^2} \int_0^l x \sin \frac{n\pi}{l} x \, dx$$

$$= \frac{200}{l^2} \left[x \left(\frac{-\cos \frac{n\pi}{l} x}{\frac{n\pi}{l}} \right) - \left(\frac{-\sin \frac{n\pi}{l} x}{\frac{n^2\pi^2}{l^2}} \right) \right]_0^l$$

$$= \frac{200}{l^2}\left[l \left(-\cos \frac{n\pi}{l} l \right) \cdot \frac{l}{n\pi} \right]$$

$$= \frac{200}{l^2}\left[\frac{-l^2}{n\pi} \cos n\pi \right]$$

$$= \frac{200}{n\pi} (-1)^{n+1} \qquad \left[\cos n\pi = (-1)^n \right]$$

$$\therefore u(x,t) = \sum_{n=1}^{\infty} \frac{200}{n\pi} (-1)^{n+1} \sin \frac{n\pi}{l} x \; e^{-\frac{\alpha^2 n^2 \pi^2 t}{l^2}}$$