STEADY STATE CONDITIONS & NON-ZERO BOUNDARY CONDITIONS:

A bar 10 cm long with insulated sides has its ends A & B kept at $20^\circ c$ & $40^\circ c$ until steady state conditions prevail. The tem at A is then suddenly raised to $50^\circ c$ and at the same instant that of B is lowered to $10^\circ c$. Find the subsequent temp. at any point of the bar at any time.

Soln: The general form of heat flow equation is $\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2}$

At steady state, $\frac{\partial u}{\partial t} = 0 \Rightarrow \frac{\partial^2 u}{\partial x^2} = 0$

$\therefore$ The general equation is $u(x) = ax + b$ — (1)

(i) $u(0) = 20$

(ii) $u(10) = 40$

Now $u(x) = ax+b$ (1)

Sub (i) in (1) $u(0) = a(0) + b$

$\boxed{20 = b}$

$\therefore u(x) = ax + 20$ — (2)

Sub (ii) in (2), $u(10) = 10a + 20$

$40 = 10a + 20$

$\boxed{2 = a}$

$\therefore u(x) = 2x + 20$

(i) $u(0) = 50$

(ii) $u(10) = 10$

Now sub (i) in (1)

$u(0) = a(0) + b$

$\boxed{50 = b}$

$\therefore u(x) = ax + 50$ — (2)

sub (ii) in (2), $u(10) = 10a + 50$

$10 = 10a + 50$

$\boxed{-4 = a}$

$\therefore u_t(x) = -4x + 50$

Since A is raised to $50^\circ C$ & B is lowered to $10^\circ C$, the steady state is changed to unsteady state.

For unsteady state,

(i) $u(0,t) = 50$

(ii) $u(10,t) = 10$

(iii) $u(x,0) = f(x) = 2x + 20$.

Suitable soln is $u(x,t) = u_t(x) + (A\cos px + B\sin px)\, e^{-\alpha^2 p^2 t}$ —— (1)

Sub B.C (i) in (1)

$u(0,t) = -4(0) + 50 + (A\cos p(0) + B\sin p(0))\, e^{-\alpha^2 p^2 t}$

$50 = -4(0) + 50 + A e^{-\alpha^2 p^2 t}$

$0 = A e^{-\alpha^2 p^2 t}$ $\quad [\because e^{-\alpha^2 p^2 t} \neq 0]$

$\boxed{0 = A}$

$\therefore u(x,t) = u_t(x) + B\sin px\, e^{-\alpha^2 p^2 t}$ —— (2)

$u(x,t) = -4x + 50 + B\sin px\, e^{-\alpha^2 p^2 t}$

Sub. B.C (ii) in (2),

$u(10,t) = u_t(x) + B\sin p(10)\, e^{-\alpha^2 p^2 t}$

$10 = -4(10) + 50 + B\sin p(10)\, e^{-\alpha^2 p^2 t}$

$10 = -40 + 50 + B\sin p(10)\, e^{-\alpha^2 p^2 t}$

$0 = B\sin p(10)\, e^{-\alpha^2 p^2 t}$

$[\because e^{-\alpha^2 p^2 t} \neq 0,\ B \neq 0]$ $\sin p(10) = 0$

$\boxed{p = \frac{n\pi}{10}}$

Sub $p = \frac{n\pi}{10}$ in ②, $u(x,t) = U_t(x) + B \sin\left(\frac{n\pi}{10}\right)x \; e^{-\frac{\alpha^2 n^2 \pi^2}{100} t}$

General soln., $u(x,t) = -4x + 50 + B \sin\left(\frac{n\pi}{10}\right)x \; e^{-\frac{\alpha^2 n^2 \pi^2}{100} t}$ ——— ③

Sub B.C (iii) in ③, $u(x,t) = -4x + 50 + \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi}{10}\right)x \; e^{-\frac{\alpha^2 n^2 \pi^2}{100} t}$

$$u(x,0) = -4x + 50 + \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi}{10}x\right) e^{-\frac{\alpha^2 n^2 \pi^2}{100}(0)}$$

$$2x + 20 = -4x + 50 + \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi}{10}x\right)$$

$$6x - 30 = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi}{10}\right)x.$$

To find B_n, $B_n = \frac{2}{10}\int_0^{10} (6x-30) \sin\left(\frac{n\pi}{10}x\right) dx.$

$$u = 6x - 30 \qquad v = \sin\left(\frac{n\pi}{10}\right)x$$

$$u' = 6 \qquad v_1 = \frac{-\cos\left(\frac{n\pi}{10}\right)x}{\frac{n\pi}{10}}$$

$$u'' = 0 \qquad v_2 = -\frac{\sin\left(\frac{n\pi}{10}\right)x}{\left(\frac{n\pi}{10}\right)^2}$$

$$B_n = \frac{1}{5}\left[(6x-30)\left(\frac{-\cos\left(\frac{n\pi}{10}\right)x}{\frac{n\pi}{10}}\right) - 6\left(-\frac{\sin\left(\frac{n\pi}{10}\right)x}{\left(\frac{n\pi}{10}\right)^2}\right)\right]_0^{10}$$

$$B_n = -\frac{60}{n\pi}\left[1 + (-1)^n\right]$$

$$\therefore u(x,t) = -4x + 50 + \sum_{n=1}^{\infty}\left[-\frac{60}{n\pi}\left(1+(-1)^n\right)\right] \sin\left(\frac{n\pi}{10}\right)x \; e^{-\frac{\alpha^2 n^2 \pi^2}{100} t}$$