

One dimensional Heat Equation [parabolic]

1DHE (Rod)

The one dimensional heat eqn is

$$\frac{\partial u}{\partial t} = a^2 \frac{\partial^2 u}{\partial x^2}$$

where $a^2 = \frac{\text{Thermal conductivity}}{\text{Density} \times \text{specific heat}}$

i.e., $a^2 = \frac{k}{\rho c}$ which is called the diffusivity.

Possible solns. of ODHE:

- 1) $u(x, t) = (A_1 e^{px} + A_2 e^{-px}) A_3 e^{a^2 p^2 t}$
- 2) $u(x, t) = (A_4 \cos px + A_5 \sin px) A_6 e^{-a^2 p^2 t}$
- 3) $u(x, t) = (A_7 x + A_8) A_9$

Sustainable soln.:

$$u(x, t) = (A \cos px + B \sin px) e^{-a^2 p^2 t}$$

Assumptions for deriving the ODHE:

- 1] Heat flows from higher to lower temperature.
- 2] The amount of heat required to produce a given temperature change in a body is proportional to the mass of the body and to the temperature change.
- 3] The rate at which heat flows through an area is proportional to the area and to the temperature gradient normal to the surface.

This constant of proportionality is known as the thermal conductivity (k) of the material. It is known as Fourier's law of heat conduction.

Steady state condition:

The state in which the temperature depends only on the distance but not on time t , is called steady state. Therefore $u(x, t)$ becomes $u(x)$ under the steady state.

Note: $u(x) = \left(\frac{b-a}{l}\right)x + a$

TYPE - I [Steady state conditions both ends at zero temperature]

J. A rod of length l has its ends A and B kept at 0°C and 100°C until steady state condition prevail. If the temperature at B is reduced suddenly to 0°C and kept so while that of A is maintained, find the temperature $u(x, t)$ at a distance x from A and at time t .

Soln =

The PDE is

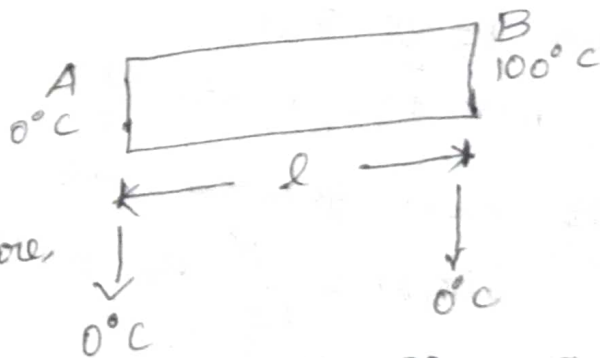
$$\frac{\partial u}{\partial t} = a^2 \frac{\partial^2 u}{\partial x^2}$$

The boundary conditions are,

i). $u(0, t) = 0$

ii). $u(l, t) = 0$

iii). $u(x, 0) = \frac{100x}{l}$



$$\begin{aligned} f(x) &= \left(\frac{b-a}{l}\right)x + a \\ &= \left(\frac{100-0}{l}\right)x + 0 \\ &= \frac{100x}{l} \end{aligned}$$

The suitable soln. is

$$u(x, t) = (A \cos px + B \sin px) e^{-a^2 p^2 t} \rightarrow (1)$$

Applying (i) in (1), we get

$$u(0, t) = 0$$

$$(A + 0) e^{-a^2 p^2 t} = 0$$

$$A e^{-a^2 p^2 t} = 0$$

Here $e^{-a^2 p^2 t} \neq 0$ (It is a fn. of 't')

$$\Rightarrow A = 0$$

Subs. $A = 0$ in (1), we get

$$u(x, t) = B \sin px e^{-a^2 p^2 t} \rightarrow (2)$$

Applying (ii) in (2), we get

$$u(l, t) = 0$$

$$B \sin pl e^{-a^2 p^2 t} = 0$$

Here $e^{-a^2 p^2 t} \neq 0$ (It is a fn. of 't')

$B \neq 0$ (suppose $B = 0$, we get a trivial soln)

$$\Rightarrow \sin pl = 0$$

$$pl = n\pi$$

$$p = \frac{n\pi}{l}$$

Subs. $p = \frac{n\pi}{l}$ in (2),

$$u(x, t) = B \sin \frac{n\pi x}{l} e^{-a^2 \frac{n^2 \pi^2}{l^2} t} \rightarrow (3)$$

The most general soln. is

$$u(x, t) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{l} e^{-a^2 \frac{n^2 \pi^2}{l^2} t} \rightarrow (4)$$

Applying (iii)

$$u(x, 0) = \frac{100x}{l}$$

$$\sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{l} = \frac{100x}{l}$$

HRSS $\sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l} = \frac{100x}{l} \quad \therefore B_n = b_n$

$$b_n = \frac{2}{l} \int_0^l f(x) \sin \frac{n\pi x}{l} dx$$

$$B_n = \frac{2}{l} \int_0^l \frac{100x}{l} \sin \frac{n\pi x}{l} dx$$

$$= \frac{200}{l^2} \int_0^l x \sin \frac{n\pi x}{l} dx$$

$$= \frac{200}{l^2} \left[x \left(\frac{-\cos \frac{n\pi x}{l}}{n\pi/l} \right) - 1 \left(\frac{-\sin \frac{n\pi x}{l}}{n^2 \pi^2 / l^2} \right) \right]_0^l$$

$$= \frac{200}{l^2} \left[-\frac{l x}{n\pi} \cos \frac{n\pi x}{l} + \frac{l^2}{n^2 \pi^2} \sin \frac{n\pi x}{l} \right]_0^l$$

$$= \frac{200}{l^2} \left[\frac{-l^2}{n\pi} (-1)^n - 0 \right]$$

$$= \frac{-200}{n\pi} (-1)^n$$

$$B_n = \frac{200 (-1)^{n+1}}{n\pi}$$

Subs. B_n value in (4), we get $\frac{200 n^2 \pi^2 \pm}{l^2}$

$$u(x, t) = \sum_{n=1}^{\infty} \frac{200 (-1)^{n+1}}{n\pi} \sin \frac{n\pi x}{l} e^{-\frac{200 n^2 \pi^2 t}{l^2}}$$

$$= \frac{200}{\pi} \sum_{n=1}^{\infty} (-1)^{n+1} \sin \frac{n\pi x}{l} e^{-\frac{200 n^2 \pi^2 t}{l^2}}$$

27. A rod 30cm long has its ends A and B kept at 20° and 80° resly. unti, steady state conditions prevail. The temperature at each end is then suddenly reduced to 0°C and kept so. Find the resulting temperature function $u(x, t)$ taking $x=0$ at A.

Soln.:

The PDE is,

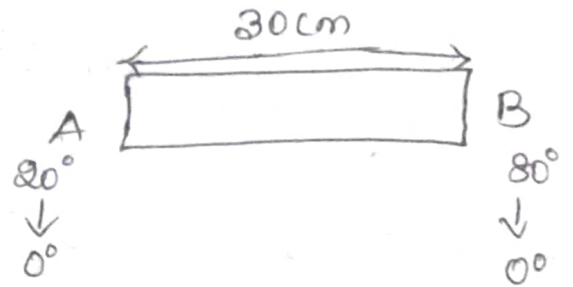
$$\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2}$$

The boundary conditions are

i). $u(0, t) = 0$

ii). $u(30, t) = 0$

iii). $u(x, 0) = 2x + 20$



$$f(x) = \left(\frac{b-a}{l}\right)x + a$$

$$= \left(\frac{80-20}{30}\right)x + 20$$

$$f(x) = 2x + 20$$

The suitable soln. is

$$u(x, t) = (A \cos px + B \sin px) e^{-\alpha^2 p^2 t} \rightarrow (1)$$

Applying (i) in (1),

$$u(0, t) = 0$$

$$(A + 0) e^{-\alpha^2 p^2 t} = 0$$

$$A e^{-\alpha^2 p^2 t} = 0$$

Here $e^{-\alpha^2 p^2 t} \neq 0$ [It is a fn. of 't']

$$\boxed{A=0}$$

Subs. $A=0$ in (1),

$$u(x, t) = B \sin px e^{-\alpha^2 p^2 t} \rightarrow (2)$$

Applying (ii) in (2),

$$u(30, t) = 0$$

$$B \sin 30p e^{-a^2 p^2 t} = 0$$

Here $e^{-a^2 p^2 t} \neq 0$ (It is a fn of 't')

and $B \neq 0$ [Suppose $B=0$, we get a trivial soln.]

$$\Rightarrow \sin 30p = 0$$

$$p = \frac{n\pi}{30}$$

Subs. $p = \frac{n\pi}{30}$ in (2),

$$e^{-a^2 \frac{n^2 \pi^2}{900} t} \rightarrow (3)$$

$$u(x, t) = B \sin \frac{n\pi x}{30} e^{-a^2 \frac{n^2 \pi^2}{900} t}$$

The most general soln. is $e^{-a^2 \frac{n^2 \pi^2}{900} t} \rightarrow (4)$

$$u(x, t) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{30} e^{-a^2 \frac{n^2 \pi^2}{900} t}$$

Applying (iii)

$$u(x, 0) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{30} = 2x + 20$$

HRSS $\sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{30} = 2x + 20 \quad \therefore B_n = b_n$

where $b_n = \frac{2}{l} \int_0^l f(x) \sin \frac{n\pi x}{l} dx$

$$\therefore B_n = \frac{2}{30} \int_0^{30} (2x + 20) \sin \frac{n\pi x}{30} dx$$

$$= \frac{1}{15} \left[(2x + 20) \left(\frac{\cos \frac{n\pi x}{30}}{n\pi/30} \right) - 2 \left(\frac{\sin \frac{n\pi x}{30}}{n^2 \pi^2 / 900} \right) \right]_0^{30}$$

$$= \frac{1}{15} \left[-\frac{30}{n\pi} (2x + 20) \cos \frac{n\pi x}{30} + 2 \frac{900}{n^2 \pi^2} \sin \frac{n\pi x}{30} \right]_0^{30}$$

$$= \frac{1}{15} \left[\left(-\frac{30(60+20)}{n\pi} (-1)^n \right) - \left(-\frac{30(0+20)}{n\pi} + 0 \right) \right]$$

$$= \frac{1}{15} \left[-\frac{2400}{n\pi} (-1)^n + \frac{600}{n\pi} \right]$$

$$= \frac{1}{15} \frac{600}{n\pi} [1 - 4(-1)^n]$$

$$B_n = \frac{40}{n\pi} [1 - 4(-1)^n]$$

$$(4) \Rightarrow u(x, t) = \sum_{n=1}^{\infty} \frac{40}{n\pi} [1 - 4(-1)^n] \sin \frac{n\pi x}{30} e^{-a^2 \frac{n^2 \pi^2}{900} t}$$