Principle moment of inertia

At a specified angle of rotation, the product of inertia of an unsymmetrical section also becomes Zero.

The perpendicular axes about which product of inertia is zero are called “Principal axes” and the Moments of inertia with respect to these are called as “principal moments of inertia”.

Among two values of moments of inertia about the set of principal axes, one will be maximum and the other will be minimum. “The maximum moment of inertia is known as major principal moment of inertia and the minimum moment of inertia is known as minor principal moment of inertia”.

Location of principal axes:

The product of inertia with the respect to the principal axes is zero.

$$\tan 2\theta = \left(\frac{-2I_{XY}}{I_{XX} - I_{YY}}\right)$$

Solving this equation, there are two values for θ, ie θ_1 and θ_2 which one 90° apart or two principal axes are perpendicular to each other .

Note: While solving if θ value is negative, the principal axis should be marked in the Clockwise direction with respect to XX axis.

Principal moments of inertia:

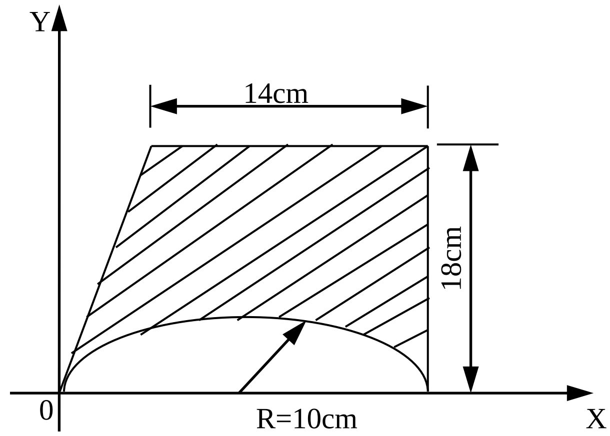
Maximum principal moments of inertia

$$I_{max} = \left(\frac{I_{XX} + I_{YY}}{2}\right) + \sqrt{\left(\frac{I_{XX} - I_{YY}}{2}\right)^2 + (I_{XY})^2}$$

Minimum principal moments of inertia

$$I_{min} = \left(\frac{I_{XX} + I_{YY}}{2}\right) - \sqrt{\left(\frac{I_{XX} - I_{YY}}{2}\right)^2 + (I_{XY})^2}$$

Problem 20: Calculate the principal moments of inertia of the section shown in fig.



Solution:

Step1: location of centroid

Portion (1) (rectangle, base 14 cm; height 18cm)

Area, $a_1 = 14 \times 18 = 252\ cm^2$

$$x_1 = (20 - 14) + \frac{14}{2} = 13\ cm\ ;\ y_1 = \frac{18}{2} = 9\ cm$$

Portion 2 (Triangle, base (20-14) cm; height 18 cm)

Area, $a_2 = \frac{1}{2} \times (20 - 14) \times 18 = 54\ cm^2$

$$x_2 = \frac{2}{3} \times (20 - 14) = 4\ cm;\ y_2 = \frac{1}{3} \times 18 = 6\ cm$$

Portion 3 (semi-circle, radius 10cm)

Area, $a_3 = \frac{1}{2} \times \pi r^2 = \frac{1}{2} \times \pi \times (10)^2 = 157.08\ cm^2$

$$x_3 = r = 10\ cm;\ \ y_3 = \frac{4r}{3\pi} = \frac{4 \times 10}{3\pi} = 4.244\ cm$$

$$\therefore \bar{x} = \frac{a_1 x_1 + a_2 x_2 - a_3 x_3}{a_1 + a_2 - a_3} = 12.9\ cm$$

$$\bar{y} = \frac{a_1 y_1 + a_2 y_2 - a_3 y_3}{a_1 + a_2 - a_3} = 12.928\ cm$$

Step2: Product of inertia co-ordinates

Portion (1) $a_1 = 252\ cm^2$

$$x'_1 = (13 - 12.9) = 0.1\ cm$$
$$y'_1 = -(12.928 - 9) = -3.928\ cm$$

Portion (2) $a_2 = 54\ cm^2$

$$x'_2 = -(12.9 - 4) = -8.9\ \ cm$$

$$y'_2 = -(12.928 - 6) = -6.928\ cm$$

Portion (3) $a_3 = 157.08\ cm^2$

$$x'_3 = -(12.9 - 10) = -2.9\ cm$$

$$y'_3 = -(\bar{y} - y_3) = -(12.928 - 4.244) = -8.684\ cm$$

Step3: Product of inertia

Product of inertia about centroidal axes

$$I_{XY} = \sum a x' y'$$

$$= a_1 x'_1 y'_1 + a_2 x'_2 y'_2 - a_3 x'_3 y'_3$$

$$= [252 \times 0.1 \times (-3.928)] + [54 \times (-8.9) \times (-6.928)] - [157.08 \times (-2.9) \times (-8.684)]$$

$$= -725.21\ cm^4$$

Step 4: moments of inertia

m.I about XX axis, $I_{XX} = (I_{XX})_1 + (I_{XX})_2 - (I_{XX})_3$

$$(I_{XX})_1 = \frac{14 \times 18^3}{12} + [(14 \times 18) \times (12.928 - 9)^2]$$

$$= 10692.15\ cm^4$$

$$(I_{XX})_2 = \frac{6 \times 18^3}{36} + \left\{\frac{1}{2} \times 6 \times 18 \times (12.928 - 6)^2\right\}$$

$$= 3563.85\ cm^4$$

$$(I_{XX})_3 = 0.0068 d^4 + \left\{\left(\frac{1}{2} \times \frac{\pi d^2}{4}\right) + (\bar{y} - y_3)^2\right\}$$

$$= (0.0068 \times 20^4) + \left\{\frac{1}{2} \times \frac{1}{4} \times \pi \times 10^2 - (12.928 - 4.244)^2\right\}$$

$$= 12933.7\ cm^4$$

$$\therefore I_{XX} = 1322.3\ cm^4$$

m.I about YY axis, $I_{YY} = (I_{YY})_1 + (I_{YY})_2 - (I_{YY})_3$

$$(I_{YY})_1 = \frac{18 \times 14^3}{12} + \left\{\frac{(18 \times 14)}{2} + (12 - 13)^2\right\}$$

$$= 4118.52\ cm^4$$

$$(I_{YY})_2 = \frac{18 \times 6^3}{36} + \left\{\frac{1}{2} \times 18 \times 6 \times (12.9 - 4)^2\right\}$$

$$= 4385.34\ cm^4$$

$$(I_{YY})_3 = \left(\frac{1}{2} \times \frac{\pi d^4}{64}\right) + \left\{\frac{1}{2} \times \frac{\pi \times 20^2}{4} \times (12.9 - 10)^2\right\}$$

$$= 5247.94\ cm^4$$

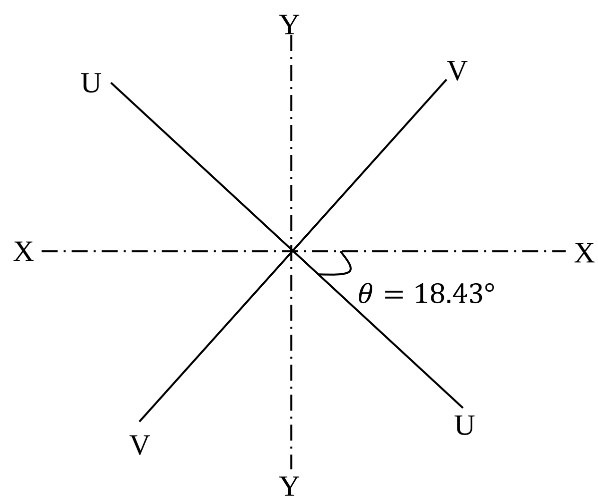
$$\therefore I_{YY} = 3255.92\ cm^4$$

Step 5: Location of principal axes

$$\tan 2\theta = \left(\frac{-2I_{XY}}{I_{XX} - I_{YY}}\right)$$

$$= \frac{-(2 \times -725.21)}{(1322.3 - 3255.92)} = -(0.75)$$

$$\theta = -18.43°$$



Step6: Principal moments if inertia

Max. principal moment of inertia

$$I_{max} = \left(\frac{I_{XX} + I_{YY}}{2}\right) + \sqrt{\left(\frac{I_{XX} - I_{YY}}{2}\right)^2 + (I_{XY})^2}$$

$$= \left(\frac{1322.3 + 3255.92}{2}\right) + \sqrt{\left(\frac{1322.3 - 3255.92}{2}\right)^2 + (-725.21)^2}$$

$$= 3497.68\ cm^4$$

Min. principal moments of inertia

$$I_{min} = \left(\frac{I_{XX} + I_{YY}}{2}\right) - \sqrt{\left(\frac{I_{XX} - I_{YY}}{2}\right)^2 + (I_{XY})^2}$$

$$= 1080.54\ cm^4$$

To check

$$I_{XX} + I_{YY} = I_{max} + I_{min}$$

$$(1322.3 + 3255.92) = (3497.68 + 1080.54)$$

$$4578.22 = 4578.22$$