UNIT-4

Applications of Partial Differential Equations

classification of II order PDE:

z - Dependent Variable

x, y - Independent Variables.

General Form:

$$A\frac{\partial^2 u}{\partial x^2} + B\frac{\partial^2 u}{\partial x \partial y} + C\frac{\partial^2 u}{\partial y^2} + f\left(x, y, u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}\right) = 0 \quad \text{— (1)}$$

A, B and C are functions of x and y.

classification: If $B^2 - 4AC < 0$, (1) is Elliptic

If $B^2 - 4AC = 0$, (1) is Parabolic

If $B^2 - 4AC > 0$, (1) is Hyperbolic.

Initial Conditions: The conditions which are defined at time $t = 0$ are called initial conditions.

Boundary Conditions: The conditions which are defined at the boundary of the region or interval are called boundary conditions.

Boundary Value Problems: The PDE's which satisfy some specified initial and boundary conditions are called as boundary Value Problems.

Important PDE's:

Wave Equation $\frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 y}{\partial x^2}$

One dimensional heat flow Equation $\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2}$

Two dimensional heat flow Equation $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$.

Problems :

1) Find the nature of the PDE

$$f_{xx} + 2f_{xy} + 4f_{yy} = 0, \quad x>0, y>0$$

Solution:

G.F: $AU_{xx} + BU_{xy} + CU_{yy} + f(x,y,u,u_x,u_y) = 0.$

Here, $A = 1$; $B = 2$; $C = 4$

check: $B^2 - 4AC = 4 - 4(1)4 = 4 - 16 = -12 = -ve$

$$B^2 - 4AC < 0$$

$\therefore$ The gn PDE is elliptic in Nature

2) Find the nature of PDE $4U_{xx} + 4U_{xy} + U_{yy} + 2U_x - U_y = 0$

Solution:

G.F: $AU_{xx} + BU_{xy} + CU_{yy} + f(x,y,u,u_x,u_y) = 0.$

Here $A = 4$; $B = 4$; $C = 1$

check: $B^2 - 4AC = 4^2 - 4(4)(1) = 16 - 16 = 0$

$$B^2 - 4AC = 0$$

$\therefore$ The given PDE is Parabolic in Nature.

3) Classify the PDE $\dfrac{\partial^2 u}{\partial x^2} = \dfrac{\partial^2 u}{\partial y^2}$

Soln:

$$\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2} = 0$$

Here $A = 1$; $B = 0$; $C = -1$

$$B^2 - 4AC = 0 - 4(1)(-1)$$
$$= 4 > 0$$

$\therefore B^2 - 4AC > 0$

$\therefore$ The given PDE is hyperbolic in Nature.

4) Classify the PDE,

$$x^2 f_{xx} + (1-y^2) f_{yy} = 0; \quad -1<y<1; \; -\infty<x<\infty$$

Soln:

$A = x^2$; $B = 0 \cancel{(1-y^2)}$; $C = \cancel{0}\, 1-y^2)$

check: $B^2 - 4AC = (1-y^2)^2 - 4(x^2)(0) = 1 - y^4 - 2y^2$

$A = x^2$; $B = 0$; $C = 1 - y^2$

$$B^2 - 4AC = 0 - 4(x^2)(1-y^2) = -4x^2(1-y^2)$$
$$= 4x^2(y^2-1)$$

In $-\infty < x < \infty$, x^2 is always +ve

In $-1 < y < 1$, $y^2 - 1$ is (-ve)

$\therefore$ $B^2 - 4AC$ is -ve

$\therefore$ It is elliptic.

If $x = 0$; $B^2 - 4AC = 0$, $\therefore$ it is parabolic.

If $y > 1$, $y < -1$ then $B^2 - 4AC > 0$, $\therefore$ it is hyperbolic.