One dimensional Wave Equation

The Vibrating String:

The equation is given by $\frac{\partial^2 y}{\partial t^2} = a^2 \cdot \frac{\partial^2 y}{\partial x^2}$

where $a^2 = \frac{T}{M} = \frac{\text{Tension}}{\text{Mass per unit length}}$

Possible Solutions are given by

$k \to (-ve)\ k = -p^2$

① $y(x,t) = (c_1 \cos px + c_2 \sin px)(c_3 \cos pat + c_4 \sin pat)$

② $y(x,t) = (c_1 e^{px} + c_2 e^{-px})(c_3 e^{pat} + c_4 e^{-pat})$ $k \to (+ve)\ k = p^2$

③ $y(x,t) = (c_1 x + c_2)(c_3 t + c_4)$ $k = 0$

Since we deal with problems on vibrations, y must be a periodic function of x and t, we choose the soln with trigonometric terms since sine and cosine are of periodic in nature.

① we consider now

Steps to solve one dimensional wave equation with zero initial velocity

Step 1 : Write the boundary conditions

a) $y(0,t) = 0 \quad \forall t$

b) $y(l,t) = 0 \quad \forall t$

c) $\frac{\partial y(x,0)}{\partial t} = 0 \quad \forall x$ ($\because$ initial velocity is zero)

d) $y(x,0) = f(x)$

Step 2 : choose the solution

$$y(x,t) = (c_1 \cos px + c_2 \sin px)(c_3 \cos pat + c_4 \sin pat)$$

Step 3 : Applying condition (a), we get $c_1 = 0$

Step 4 : Applying condition (b), we get $p = \frac{n\pi}{l}$

Step 5 : Applying condition (c), we get $c_4 = 0$.

Step 6 : The general solution is

$$y(x,t) = \sum_{n=1}^{\infty} c_n \sin \frac{n\pi x}{l} \cos \frac{n\pi at}{l}$$

Step 7 : Applying condition (d),

$$y(x,0) = \sum_{n=1}^{\infty} c_n \sin \frac{n\pi x}{l} = f(x) \quad \text{—— (A)}$$

Expand $f(x)$ in a sine Series in $(0,l)$

$$f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l} \quad \text{—— (B)}$$

where $c_n = b_n = \frac{2}{l}\int_0^l f(x) \sin \frac{n\pi x}{l}\, dx$

Step 8 : Substituting c_n in ~~(A)~~ step 6 we get solution of the given problem.