Problem 1: (14) A string is stretched and fastened at two points $x=0$ and $x=l$ apart. Motion is started by displacing the string into the form $y=k(lx-x^2)$ from which it is released at time $t=0$. Find the displacement of any point on the string at a distance of x from one end at time t.

Solution:

The wave equation is $\dfrac{\partial^2 y}{\partial t^2} = a^2 \dfrac{\partial^2 y}{\partial x^2}$

Step 1: Boundary conditions:

a) $y(0,t)=0 \quad \forall\, t>0$

b) $y(l,t)=0 \quad \forall\, t>0$

c) $\dfrac{\partial y(x,0)}{\partial t}=0$ ($\because$ initial velocity is zero)

d) $y(x,0)=k(lx-x^2)$

Step 2: choose the solution

The solution which satisfies the boundary conditions ~~are~~ is

$$y(x,t)=(c_1\cos px + c_2 \sin px)(c_3\cos pat + c_4 \sin pat) \quad (1)$$

Step 3: Applying (a) in (1)

$y(0,t)=0 \qquad$ sub $x=0$; $t=t$

$$y(0,t)=(c_1\cos 0 + c_2\sin 0)(c_3\cos pat + c_4\sin pat)=0$$

$$\Rightarrow (c_1(1)+c_2(0))(c_3\cos pat + c_4\sin pat)=0.$$

$$c_1(c_3\cos pat + c_4\sin pat)=0.$$

$$c_3\cos pat + c_4 \sin pat \neq 0$$

$$c_1=0 \text{ and } c_3\cos pat + c_4\sin pat$$

put $c_1=0$ in (1)

$$y(x,t)= c_2\sin px\,(c_3\cos pat + c_4\sin pat) \quad (2)$$

Step 4: Applying b) in (2)

$$y(l,t)=0 \qquad x=l\ ;\ t=t$$

$$C_2 \sin pl\,(C_3 \cos pat + C_4 \sin pat) = 0$$

$$\Rightarrow C_2 \sin pl = 0 \text{ and } C_3 \cos pat + C_4 \sin pat \neq 0$$

($\because$ it is defined for all t)

Either $C_2=0$ (or) $\sin pl=0$

If $C_2=0$, we get a trivial solution

$\therefore$ We choose $\sin pl = 0$

$$pl = n\pi \qquad (\because \sin n\pi = 0)$$

$$\boxed{p = \frac{n\pi}{l}} \qquad n \text{ being an integer}$$

Sub. $p=\frac{n\pi}{l}$ in (2)

$$y(x,t) = C_2 \sin \frac{n\pi}{l}x \left(C_3 \cos \frac{n\pi}{l}at + C_4 \sin \frac{n\pi}{l}at\right) \quad (3)$$

Step 5: Applying c) in (1)

diff (3) w.r. to 't' partially,

$$\frac{\partial y}{\partial t}(x,t) = C_2 \sin \frac{n\pi}{l}x \left(-C_3\left(\frac{n\pi}{l}a\right)\sin \frac{n\pi}{l}at + C_4\left(\frac{n\pi}{l}a\right)\cos \frac{n\pi}{l}at\right)$$

Now, $\frac{\partial y}{\partial t}(x,0) = 0$

$$C_2 \sin \frac{n\pi}{l}x \left(-C_3\left(\frac{n\pi}{l}a\right)\sin \frac{n\pi}{l}a0 + C_4\left(\frac{n\pi}{l}a\right)\cos \frac{n\pi}{l}a0\right) = 0$$

$$c_2 \sin \frac{n\pi}{l} x \left(c_4 \frac{n\pi a}{l} \cos(0) \right) = 0$$

$$c_2 \sin \frac{n\pi x}{l} \left(c_4 \frac{n\pi a}{l} \right) = 0$$

$\because c_2 \neq 0$ and $\sin \frac{n\pi x}{l} \neq 0$ ($\because$ it is defined for all x

$$\Rightarrow \frac{n\pi x}{l} \neq 0$$

$$\therefore c_4 \frac{n\pi a}{l} = 0 \Rightarrow \boxed{c_4 = 0}$$

Step 6:

Substituting $c_4 = 0$ in ③ we get

$$y(x,t) = c_2 c_3 \sin \frac{n\pi x}{l} \cos \frac{n\pi a t}{l}$$

$$y(x,t) = C_n \sin \frac{n\pi x}{l} \cos \frac{n\pi a t}{l} \quad \text{where } C_n = c_2 c_3 \qquad \text{④}$$

$\therefore$ The most general solution of ④ is

$$y(x,t) = \sum_{n=1}^{\infty} C_n \sin \frac{n\pi x}{l} \cos \frac{n\pi a t}{l} \qquad \text{⑤}$$

Step 7:

Applying the boundary conditions (d) in ⑤ we get

$$y(x,0) = \sum_{n=1}^{\infty} C_n \sin \frac{n\pi x}{l} \cos(0) = k(lx - x^2)$$

$$\Rightarrow \sum_{n=1}^{\infty} C_n \sin \frac{n\pi x}{l} = k(lx - x^2) \qquad \text{⑥}$$

To find C_n expand $k(lx - x^2)$ in a half-range Fourier Sine Series in the interval $(0, l)$

$$k(lx - x^2) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l} \qquad \text{⑦}$$

where $b_n = \frac{2}{l} \int_0^l f(x) \sin \frac{n\pi x}{l} \, dx$

from half-range Fourier Sine Series

from ⑥ and ⑦ we have $C_n = b_n$

$$\therefore \quad C_n = \frac{2}{l}\int_0^l k(lx - x^2)\sin\frac{n\pi x}{l}\,dx$$

$$= \frac{2k}{l}\int_0^l (lx - x^2)\sin\frac{n\pi x}{l}\,dx$$

$u = lx - x^2$ $\qquad\qquad$ $dv = \sin\frac{n\pi x}{l}\,dx$

$u' = l - 2x$ $\qquad\qquad$ $v = -\cos\left(\frac{n\pi x}{l}\right)\cdot\frac{1}{\left(\frac{n\pi}{l}\right)} = \frac{-l}{n\pi}\cos\left(\frac{n\pi x}{l}\right)$

$u'' = -2$ $\qquad\qquad$ $v_1 = \frac{-l^2}{n^2\pi^2}\sin\left(\frac{n\pi x}{l}\right)$

$u''' = 0$ $\qquad\qquad$ $v_2 = \frac{+l^3}{n^3\pi^3}\cos\left(\frac{n\pi x}{l}\right)$

$$C_n = \frac{2k}{l}\left[(lx - x^2)\left(\frac{-l}{n\pi}\cos\left(\frac{n\pi x}{l}\right)\right) - (l-2x)\left(\frac{-l^2}{n^2\pi^2}\sin\left(\frac{n\pi x}{l}\right)\right) + (-2)\left(\frac{+l^3}{n^3\pi^3}\cos\left(\frac{n\pi x}{l}\right)\right)\right]_0^l$$

$$= \frac{2k}{l}\left[\left(0 + 0 - \frac{2l^3}{n^3\pi^3}\cos(n\pi)\right) - \left(0 + 0 - \frac{2l^3}{n^3\pi^3}\cos(0)\right)\right]$$

$$= \frac{2k}{l}\left[\frac{+2l^3}{n^3\pi^3}\left[-(-1)^n + 1\right]\right]$$

$$= \frac{2k\,2l^3}{l\,n^3\pi^3}\left[1 - (-1)^n\right]$$

$$C_n = \frac{4kl^2}{n^3\pi^3}\left[1 - (-1)^n\right]$$

$$\therefore C_n = \begin{cases} \frac{8kl^2}{n^3\pi^3}, & \text{if } n \text{ is odd} \\ 0, & \text{if } n \text{ is even.} \end{cases}$$

Substituting C_n in (5), we get

$$y(x,t) = \sum_{n=1,3,5}^{\infty} \frac{8kl^2}{n^3\pi^3}\sin\frac{n\pi x}{l}\cos\frac{n\pi at}{l}$$

$$y(x,t) = \frac{8kl^2}{\pi^3}\sum_{n=1,3,5}^{\infty}\frac{1}{n^3}\sin\frac{n\pi x}{l}\cos\frac{n\pi at}{l}$$

Problem-2: A tightly stretched flexible string has its ends fixed at $x=0$ and $x=l$. At time $t=0$, the string is given a shape defined by $f(x)=kx^2(l-x)$, where k is a constant and then released from rest. Find the displacement of any point x of the string at any time $t>0$.

Solution: Same as problem 1 till step 6

Step 7:

Applying (d) in 5 we get

$$y(x,0)=\sum_{n=1}^{\infty} c_n \sin\frac{n\pi x}{l} = kx^2(l-x) \quad \text{——— (6)}$$

To find c_n expand $kx^2(l-x)$ in a half-range fourier sine series in $(0,l)$.

$$kx^2(l-x)=\sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l} \quad \text{——— (7)}$$

where $b_n=\frac{2}{l}\int_0^l f(x)\sin\left(\frac{n\pi x}{l}\right)dx$.

From 6 & 7 we have

$$\boxed{c_n=b_n}$$

$$\therefore c_n=\frac{2}{l}\int_0^l kx^2(l-x)\sin\frac{n\pi x}{l}\,dx$$

$$=\frac{2k}{l}\int_0^l x^2(l-x)\sin\frac{n\pi x}{l}\,dx$$

$u=x^2(l-x)=x^2l-x^3$ $\quad dv=\sin\frac{n\pi x}{l}\,dx$

$u'=2xl-3x^2$ $\quad v=\frac{-l}{n\pi}\cos\left(\frac{n\pi x}{l}\right)$

$u''=2l-6x$ $\quad v_1=\frac{-l^2}{n^2\pi^2}\sin\left(\frac{n\pi x}{l}\right)$

$u'''=-6$ $\quad v_2=\frac{+l^3}{n^3\pi^3}\cos\left(\frac{n\pi x}{l}\right)$

$v_3=\frac{l^4}{n^4\pi^4}\sin\left(\frac{n\pi x}{l}\right)$

$$\therefore\ C_n = \frac{2k}{l}\left[\left(lx^2-x^3\right)\left(\frac{-l}{n\pi}\cos\frac{n\pi x}{l}\right)-\left(2lx-3x^2\right)\left(\frac{-l^2}{n^2\pi^2}\sin\frac{n\pi x}{l}\right)\right.$$

$$\left.+\left(2l-6x\right)\left(\frac{l^3}{n^3\pi^3}\cos\frac{n\pi x}{l}\right)-(-6)\left(\frac{l^4}{n^4\pi^4}\sin\frac{n\pi x}{l}\right)\right]_0^l$$

$$C_n = \frac{2k}{l}\left[\left(0-0-4l\left(\frac{l^3}{n^3\pi^3}(-1)^n\right)+0\right)\right.$$

$$\left.-\left(0-0+2l\left(\frac{l^3}{n^3\pi^3}\right)\cdot(1)\right)\right]$$

$$= \frac{2k}{l}\left[\frac{l^3}{n^3\pi^3}\left(-4l(-1)^n-2l\right)\right]$$

$$C_n = \frac{-2kl^3\,2l}{l\,n^3\pi^3}\left[2(-1)^n+1\right] \quad \text{——— (8)}$$

Substituting C_n (8) in (5) we get

$$y(x,t) = \sum_{n=1}^{\infty}\left(\frac{-4kl^3}{n^3\pi^3}\right)\left(2(-1)^n+1\right)\sin\frac{n\pi x}{l}\cos\frac{n\pi at}{l}$$

$$y(x,t) = -\frac{4kl^3}{\pi^3}\sum_{n=1}^{\infty}\frac{1}{n^3}\left(1+2(-1)^n\right)\sin\frac{n\pi x}{l}\cos\frac{n\pi at}{l}$$

↑ is the solution,

3.) A tightly stretched string with fixed end points $x=0$ and $x=l$ is initially in a position given by $y(x,0) = y_0 \sin^3 \frac{\pi x}{l}$. If it is released from rest from this position, find the displacement y at any distance x from one end at any time t.

Solution: Same till step 6. $y(x,t) = \sum_{n=1}^{\infty} C_n \sin \frac{n\pi x}{l} \cos \frac{n\pi a t}{l}$.

Step 7: Applying (d) in (5) we get,

$$y(x,0) = \sum_{n=1}^{\infty} C_n \sin \frac{n\pi x}{l} = y_0 \sin^3 \frac{\pi x}{l} \quad \text{——— (6)}$$

We know that, $\sin^3\theta = \frac{1}{4}(3\sin\theta - \sin 3\theta)$

$$y_0 \sin^3 \frac{\pi x}{l} = \sum_{n=1}^{\infty} C_n \sin \frac{n\pi x}{l}$$

$$y_0\left[\frac{1}{4} 3\sin\left(\frac{\pi x}{l}\right) - \sin \frac{3\pi x}{l}\right] = \sum_{n=1}^{\infty} C_n \sin \frac{n\pi x}{l}$$

$$\frac{3y_0}{4}\sin\frac{\pi x}{l} - \frac{y_0}{4}\sin\frac{3\pi x}{l} = C_1 \sin\frac{\pi x}{l} + C_2\sin\frac{2\pi x}{l} + C_3 \sin\frac{3\pi x}{l}$$

On comparing the coefficients on either side.

$$C_1 = \frac{3y_0}{4}\ ;\ C_2 = 0\ ;\ C_3 = -\frac{y_0}{4}\ ;\ C_4 = C_5 = C_6 = 0.$$

Sub C_1, C_2, C_3 in (5)

$$\cancel{\therefore y(x,t) = \frac{3y_0}{4}\sin\frac{\pi x}{l} + 0 + \left(-\frac{y_0}{4}\right)\sin\frac{3\pi x}{l}}.$$

$$y(x,t) = \frac{3y_0}{4}\sin\frac{\pi x}{l}\cos\frac{\pi a t}{l} - \frac{y_0}{4}\sin\frac{3\pi x}{l}\cos\frac{3\pi a t}{l}$$

is the solution of the gn qn.