One-dimensional Heat Flow Equation:

$$\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2}$$

The solutions are

(i) $u(x,t) = c_1(c_2 x + c_3)$

(ii) $u(x,t) = c_3 e^{\alpha^2 p^2 t}(c_4 e^{px} + c_5 e^{-px})$

(iii) $u(x,t) = c_6 e^{-\alpha^2 p^2 t}(c_7 \cos px + c_8 \sin px)$

(or) $u(x,t) = e^{-\alpha^2 p^2 t}(A \cos px + B \sin px)$

where $A = c_6 c_7$ and $B = c_6 c_8$.

The most suitable solution is

$$u(x,t) = e^{-\alpha^2 p^2 t}(A \cos px + B \sin px).$$

Problem: 1 Solve the eqn $\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2}$ subject to the boundary conditions

$u(0,t) = 0$ [Both ends are at 0 temperature

$u(l,t) = 0$

$u(x,0) = x$.

Solution:

The one dimensional Heat flow eqn is (given)

$$\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2}$$

Step 1: Boundary conditions are

(a) $u(0,t) = 0 \quad \forall\, t > 0$

(b) $u(l,t) = 0 \quad \forall\, t > 0$

(c) $u(x,0) = x \quad \forall\, x$ in $(0,l)$

Step 2: ~~Applying (a) in~~ The correct solution is

$$u(x,t) = (A\cos px + B\sin px)\, e^{-\alpha^2 p^2 t} \quad \text{——— ①}$$

Step 3: Apply ⓐ in ①

$$u(0,t) = (A\cos 0 + B\sin 0)\, e^{-\alpha^2 p^2 t} = 0$$

$$A\, e^{-\alpha^2 p^2 t} = 0$$

$\Rightarrow A = 0$ and $e^{-\alpha^2 p^2 t} \neq 0$ (since it is defined $\forall\, t$)

Sub $A = 0$ in ①, we get

$$u(x,t) = B\sin px\, e^{-\alpha^2 p^2 t} \quad \text{——— ②}$$

Step 4: Apply ⓑ in ②

$$u(l,t) = B\sin pl\, e^{-\alpha^2 p^2 t} = 0.$$

~~For~~ $\Rightarrow B \neq 0$ (since $B = 0$ is a trivial solution)

$\sin pl = 0$ and $e^{-\alpha^2 p^2 t} \neq 0$ (since it is defined $\forall\, t$)

$$\Rightarrow p = \frac{n\pi}{l}$$

sub $p = \frac{n\pi}{l}$ in ② we get

$$u(x,t) = B\sin\frac{n\pi x}{l}\, e^{-\frac{\alpha^2 n^2 \pi^2 t}{l^2}} \quad \text{——— ③}$$

∴ The most general solution is

$$u(x,t) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{l}\, e^{-\frac{\alpha^2 n^2 \pi^2 t}{l^2}} \quad \text{——— ④}$$

Step 5: Apply (c) in (3) we get

$$u(x,0) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{l} e^{0} = x.$$

$$\Rightarrow \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{l} = x \quad \text{—— (5)}$$

To find B_n expand x in $(0,l)$ in a half-range Fourier Sine Series

$$x = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l} \quad \text{—— (6)}$$

From (5) and (6) we have $\boxed{b_n = B_n}$

$$\therefore B_n = b_n = \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}\,dx$$

$$= \frac{2}{l}\int_0^l x \sin\frac{n\pi x}{l}\,dx$$

$$= \frac{2}{l}\left[x\left(\frac{-l}{n\pi}\cos\frac{n\pi x}{l}\right) + \frac{l^2}{n^2\pi^2}\sin\frac{n\pi x}{l}\right]_0^l$$

$$= \frac{2}{l}\left[\frac{-l^2}{n\pi}(-1)^n + 0 - 0 - 0\right] = \frac{-2}{l}\frac{l^2}{n\pi}(-1)^n$$

$$\therefore B_n = \frac{2l}{n\pi}(-1)^{n+1}$$

Step 6: Sub B_n in (4).

$$u(x,t) = \sum_{n=1}^{\infty} \frac{2l}{n\pi}(-1)^{n+1} \sin\frac{n\pi x}{l} e^{-\frac{\alpha^2 n^2\pi^2 t}{l^2}}.$$

$\therefore$ The soln is

$$u(x,t) = \frac{2l}{\pi}\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin\frac{n\pi x}{l} e^{-\frac{\alpha^2 n^2\pi^2 t}{l^2}}$$

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