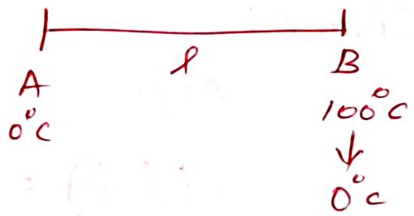
Steady State Conditions:

The state in which the temperature does not vary with respect to time 't' is called steady state. Therefore when steady state condition exists, $u(x,t)$ becomes $u(x)$.

Problems – S S conditions when both ends at '0' Temperature:

1) A rod of length l has its ends A and B kept at 0°c and 100°c until steady state Condition prevail. If the temperature at B is reduced suddenly to 0°c and kept so while that of A is maintained, find the temperature $u(x,t)$ at a distance x from A and at time 't'.

Solution:



Step 0: The one dimensional heat flow equation is

$$\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2} \quad \text{——— } (*)$$

Step 1: The boundary conditions are

(a) $u(0,t) = 0$

(b) $u(l,t) = 0$

(c) $u(x,0) = f(x)$

Here $f(x) = \left(\frac{b-a}{l}\right)x + a$

a – A's Initial temp
b – B's Initial temp

$$f(x) = \left(\frac{100-0}{l}\right)x + 0 = \frac{100x}{l}.$$

$$\therefore f(x) = \frac{100x}{l}$$

$$\therefore u(x,0) = \frac{100x}{l}.$$

Step 2: The most suitable solution is

$$u(x,t) = (A_1 \cos px + B_2 \sin px)\, e^{-\alpha^2 p^2 t} \quad \text{——} (1)$$

Step 3: Apply (a) in (1)

$$u(0,t) = 0$$

put $x = 0$; $t = t$

$$u(x,0) = (A_1 \cos 0 + A_2 \sin 0)\, e^{-\alpha^2 p^2 t} = 0$$

$$\Rightarrow A_1 e^{-\alpha^2 p^2 t} = 0$$

$$\Rightarrow A_1 = 0 \text{ and } e^{-\alpha^2 p^2 t} \neq 0$$

$\therefore$ $A_1 = 0$ (choose)

sub $A_1 = 0$ in (1).

$$u(x,t) = (B_2 \sin px)\, e^{-\alpha^2 p^2 t} \quad \text{——} (2).$$

Step 4:

Apply (b) in (2)

$$u(l,t) = 0$$

put $x = l$ and $t = t$

$$u(l,t) = (C_2 \sin pl)\, e^{-\alpha^2 p^2 t} = 0.$$

$$C_2 \neq 0 \quad \sin pl = 0 \text{ and } e^{-\alpha^2 p^2 t} \neq 0.$$

$$\therefore pl = n\pi$$

$$\boxed{p = \frac{n\pi}{l}}$$

sub p in (2)

$$u(x,t) = B_2 \sin \frac{n\pi x}{l}\, e^{-\alpha^2 \frac{n^2\pi^2}{l^2} t} \quad \text{——} (3).$$

Step 5:

$\therefore$ The most general solution is

$$u(x,t) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{l}\, e^{-\alpha^2 \frac{n^2\pi^2}{l^2} t} \quad \text{——}$$

Step 6: Apply (c) in (4).

$$u(x,0) = f(x) = \frac{100x}{l}$$

$$u(x,0) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{l} e^{0} = \frac{100x}{l} \quad (\because B_n = b_n)$$

$$\Rightarrow \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{l} = \frac{100x}{l} \quad \text{——— (5)}$$

To find B_n

Expand $\frac{100x}{l}$ in $(0,l)$ in a half-range Fourier Sine Series

$$\therefore f(x) = \frac{100x}{l} = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l} \quad \text{——— (6)}$$

$$\text{where } b_n = \frac{2}{l}\int_0^l f(x) \sin \frac{n\pi x}{x}.$$

From (5) & (6), we have $\boxed{B_n = b_n}$

$$\therefore B_n = b_n = \frac{2}{l}\int_0^l \frac{100x}{l} \sin \frac{n\pi x}{l}\, dx.$$

$$= \frac{2\times 100}{l}\int_0^l \frac{x}{l} \sin \frac{n\pi x}{l}\, dx$$

$$= \frac{200}{l}\left[\frac{x}{l}\left(\frac{-l}{n\pi}\right)\cos\frac{n\pi x}{l} - \frac{1}{l}\left(\frac{-l^2}{n^2\pi^2}\right)\sin\frac{n\pi x}{l}\right]_0^l$$

$$= \frac{200}{l}\left[\left(\frac{-l}{n\pi}(-1)^n - 0\right) - (0)\right]$$

$$\therefore B_n = \frac{200(-1)^{n+1}}{n\pi} \quad \text{——— (7)}$$

Step 7

Sub B_n in (4).

we get

$$u(x,t) = \sum_{n=1}^{\infty} \frac{200(-1)^{n+1}}{n\pi} \sin\frac{n\pi x}{l} e^{-\alpha^2 \frac{n^2\pi^2}{l^2} t}$$

$$= \frac{200}{\pi}\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin\frac{n\pi x}{l} e^{-\alpha^2 \frac{n^2\pi^2}{l^2} t}$$

2) A rod of length 20 cm has its ends A and B kept at temperature 30°c and 90°c respectively until steady state conditions prevail. If the temp at each end is then suddenly reduced to 0°c and maintained so, find the temp. distribution at a distance 'x' from A at time 't'.

Soln: Here length $l = 20$ cm.

The heat flow eqn is $\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2}$

Step 1: The boundary conditions are

a) $u(0,t) = 0$ b) $u(l,t) = 0$ c) $u(x,0) = f(x)$

$$f(x) = \left(\frac{b-a}{l}\right)x + a$$

Here $a = 30°c$
$b = 90°c$
$l = 20$

$$= \left(\frac{90° - 30°}{20}\right)x + 30°$$

$$= \frac{60° x}{20} + 30 \quad = 3x + 30$$

c) $u(x,0) = 3(x+10)$

Step 2:

The most possible soln is

$$u(x,t) = (A\cos px + B\sin px)\, e^{-\alpha^2 p^2 t} \quad \text{—— (1)}$$

Step 3 & step 4

Applying (a) and (b) we have

Step 5:

The most general solution is

$$u(x,t) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{l}\, e^{-\alpha^2 \frac{n^2\pi^2}{l^2} t} \quad \text{—— (5)}$$

Step 6: $u(x,0) = 3(x+10)$

$$u(x,0) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{l} = 3(x+10) \quad \text{—— (6)}$$

To find B_n, expand $f(x) = 3(x+10)$ in a fourier half range Sine Series in $(0,20) = (0,l)$

$$3(x+10) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{20} \quad \text{—— (7)}$$

from (6) & (7)

$$B_n = b_n$$

$$B_n = \frac{2}{l}\int_0^l f(x) \sin \frac{n\pi x}{l}\,dx = \frac{2}{20}\int_0^{20} 3(x+10) \sin \frac{n\pi x}{20}\,dx$$

$$= \frac{3}{10}\left[(x+10)\left(\frac{-20}{n\pi}\right)\cos\frac{n\pi x}{20} - \left(\frac{-20^2}{n^2\pi^2}\right)\sin\frac{n\pi x}{20}\right]_0^{20}$$

$$= \frac{3}{10}\left[\left(-\frac{600}{n\pi}(-1)^n + \frac{400}{n^2\pi^2}(0)\right) - \left(-\frac{200}{n\pi}\right)\right]$$

$$= -\frac{1800}{10n\pi}(-1)^n + \frac{600}{10n\pi} = \frac{60}{n\pi}\left[3(-1)^{n+1}+1\right]$$

$$\therefore B_n = \frac{60}{n\pi}\left[3(-1)^{n+1}+1\right] \quad \text{—— (8)}$$

Step 7:

sub (8) in (4)

$$u(x,t) = \sum_{n=1}^{\infty} \frac{60}{n\pi}\left(3(-1)^{n+1}+1\right) \sin\frac{n\pi x}{20}\, e^{-\frac{\alpha^2\pi^2 n^2}{400}t}$$

$$u(x,t) = \frac{60}{\pi}\sum_{n=1}^{\infty} \frac{\left(3(-1)^{n+1}+1\right)}{n} \sin\frac{n\pi x}{20}\, e^{-\frac{\alpha^2\pi^2 n^2}{400}t}$$