Problems on z - Transforms :

1) $z\left[\frac{1}{n+1}\right] = \sum_{n=0}^{\infty} \frac{1}{n+1} z^{-n} = 1 + \frac{1}{2} z^{-1} + \frac{1}{3} z^{-2} + \dots$

$= 1 + \frac{(1/z)}{2} + \frac{(1/z)^2}{3} + \frac{(1/z)^3}{4} + \dots$

× and ÷ by z

$= z\left[\frac{1}{z} + \frac{(1/z)^2}{2} + \frac{(1/z)^3}{3} + \frac{(1/z)^4}{4} + \dots\right]$

$= z\left[-\log\left(1-\frac{1}{z}\right)\right]$

$= -z\log\left(1-\frac{1}{z}\right) = z\log\left(1-\frac{1}{z}\right)^{-1}$

$z\left[\frac{1}{n+1}\right] = z\log\left(\frac{z}{z-1}\right)$ //

2) $z\left[\frac{1}{n+2}\right] = \sum_{n=0}^{\infty} \frac{1}{n+2} z^{-n} = \frac{1}{2} + \frac{1}{3} z^{-1} + \frac{1}{4} z^{-2} + \dots$

$= z^2\left[\frac{(1/z)^2}{2} + \frac{(1/z)^3}{3} + \dots\right.$

$= z^2\left[-\frac{1}{z} + \frac{1}{z} + \frac{(1/z)^2}{2} + \frac{(1/z)^3}{3} + \dots\right]$

$= z^2\left[-\frac{1}{z} - \log\left(1-\frac{1}{z}\right)\right]$

$= -z - z^2\log\left(1-\frac{1}{z}\right)$

$= -z + z^2\log\left(\frac{z}{z-1}\right)$

$z\left[\frac{1}{n+2}\right] = z^2\log\left(\frac{z}{z-1}\right) - z$

$z\left[\frac{1}{n+2}\right] = z^2\log\left(\frac{z}{z-1}\right) - z$ //

3) Find $Z\left[\frac{1}{n-1}\right]$, $n>1$

$$Z\left[\frac{1}{n-1}\right] = \sum_{n=0}^{\infty} \frac{1}{n-1} z^{-n}$$

$$= \sum_{n=2}^{\infty} \frac{1}{n-1} z^{-n} = \frac{1}{z^2} + \frac{1}{2z^3} + \frac{1}{3z^4} + \cdots$$

$$= \frac{1}{z}\left[\frac{1}{z} + \frac{(1/z)^2}{2} + \frac{(1/z)^3}{3} + \cdots\right]$$

$$= \frac{1}{z}\left[-\log\left(1-\frac{1}{z}\right)\right]$$

$$= \frac{1}{z}\left[\log\left(\frac{z}{z-1}\right)\right]$$

$$Z\left[\frac{1}{n-1}\right] = \frac{1}{z}\log\left(\frac{z}{z-1}\right)$$

4) Find $Z\left[\frac{1}{(n+1)!}\right]$.

$$Z\left[\frac{1}{(n+1)!}\right] = \sum_{n=0}^{\infty} \frac{1}{(n+1)!} z^{-n} = \sum_{n=0}^{\infty} \frac{1}{(n+1)!}\left(\frac{1}{z}\right)^n$$

$$= 1 + \frac{1}{2!} \cdot \frac{1}{z} + \frac{1}{3!}\frac{1}{z^2} + \cdots$$

$$= z\left[1+\frac{1}{z} + \frac{(1/z)^2}{2!} + \frac{(1/z)^3}{3!} + \cdots - 1\right]$$

$$= z\left[e^{1/z} - 1\right]$$ (∵ Add & sub 1)

$$\therefore Z\left[\frac{1}{(n+1)!}\right] = z\left[e^{1/z} - 1\right] \,//$$

5) $Z\left[e^{-an}\right]$ $\qquad Z[a^n] = \frac{z}{z-a}$

$$Z\left[e^{-an}\right] = Z\left[(e^{-a})^n\right] = \frac{z}{z-e^{-a}} \,//$$

1) Find the z-transform of the following

(i) $\frac{1}{(n+1)(n+2)}, n > 0$ (ii) $\frac{2n+3}{(n+1)(n+2)}$

Soln:

$$\frac{1}{(n+1)(n+2)} = \frac{A}{(n+1)} + \frac{B}{(n+2)}$$

Using Partial fractions

$$1 = A(n+2) + B(n+1)$$

Put $n=-1$ $A = 1$

put $n=-2$ $B = -1$

$$\therefore \frac{1}{(n+1)(n+2)} = \frac{1}{(n+1)} - \frac{1}{(n+2)}$$

$$z\left[\frac{1}{(n+1)(n+2)}\right] = z\left[\frac{1}{(n+1)}\right] - z\left[\frac{1}{(n+2}\right]$$

W.K.T $z\left[\frac{1}{n+1}\right] = z\log\left(\frac{z}{z-1}\right)$ and

$$z\left[\frac{1}{n+2}\right] = z^2\log\left(\frac{z}{z-1}\right) - z$$

$$z\left[\frac{1}{(n+1)(n+2)}\right] = z\log\left(\frac{z}{z-1}\right) - z^2\log\left(\frac{z}{z-1}\right) + z.$$

2) $z\left[\frac{2n+3}{(n+1)(n+2)}\right]$

Using Partial Fractions,

$$\frac{2n+3}{(n+1)(n+2)} = \frac{A}{(n+1)} + \frac{B}{(n+2)}$$

$$\Rightarrow 2n+3 = A(n+2) + B(n+1)$$

put $n=-1$ $1 = A$

put $n=-2$ $-1 = -B \Rightarrow B = 1$

$$\frac{2n+3}{(n+1)(n+2)} = \frac{1}{(n+1)} + \frac{1}{(n+2)}$$

$$z\left[\frac{2n+3}{(n+1)(n+2)}\right] = z\left[\frac{1}{n+1}\right] + z\left[\frac{1}{n+2}\right]$$

Using Result $Z\left[\frac{1}{n+1}\right] = z\log\left(\frac{z}{z-1}\right)$

$$Z\left[\frac{1}{n+2}\right] = z^2\log\left(\frac{z}{z-1}\right) - z$$

$$Z\left[\frac{2n+3}{(n+1)(n+2)}\right] = z\log\left(\frac{z}{z-1}\right) + z^2\log\left(\frac{z}{z-1}\right) - z$$

3) Find $Z\left[4\cdot 3^n + 2(-1)^n\right]$

$$Z\left[4\cdot 3^n + 2(-1)^n\right] = 4Z\left[3^n\right] + 2Z\left[(-1)^n\right]$$

$$= 4\cdot\frac{z}{z-3} + 2\frac{z}{z-(-1)}$$

$$= 4\frac{z}{z-3} + 2\frac{z}{z+1}.$$

Replace t by nT.

Defn: $Z[f(t)] = \sum_{n=0}^{\infty} f(nT)z^{-n}$

1) Find $Z[t]$

$$Z[t] = \sum_{n=0}^{\infty} nT z^{-n}$$

$$= T\sum_{n=0}^{\infty} n z^{-n}$$

$$= T\, Z[n]$$

$$\because Z[n] = \frac{z}{(z-1)^2}$$

$$Z[t] = T\frac{z}{(z-1)^2}$$

2) $Z\left[e^{at}\right] = \sum_{n=0}^{\infty} e^{anT} z^{-n} = \sum_{n=0}^{\infty} \left(e^{aT}\right)^n z^{-n}$

$$= Z\left[\left(e^{aT}\right)^n\right]$$

$$= \frac{z}{z-e^{aT}}$$

3) $Z\left[e^{-at}\right] = \frac{z}{z-e^{-aT}}$

4) $Z[\sin at]$ and $Z[\cos at]$ put $t = nT$ and use previous r^n formula