Convolution of Two Sequences

The Convolution of two sequences $\{f(n)\}$ and $\{g(n)\}$ is defined as

$$f(n) * g(n) = \sum_{r=0}^{n} f(r)\, g(n-r) \quad \text{(for right-sided sequence)}$$

$$f(n) * g(n) = \sum_{r=-\infty}^{\infty} f(r)\, g(n-r) \quad \text{(for two-sided or bilateral sequence)}$$

The convolution of two functions $f(t)$ and $g(t)$ is defined as

$$f(t) * g(t) = \sum_{r=0}^{n} f(rT)\, g[(n-r)T],$$

where T is the sampling period.

Convolution Theorem for Z-Transform

(i) $Z[f(n) * g(n)] = F(z) \cdot G(z)$

(ii) $Z[f(t) * g(t)] = F(z) \cdot G(z)$

Problems:

1) Using Convolution Theorem find $Z^{-1}\left[\dfrac{z^2}{(z-4)(z-3)}\right]$.

Soln:

$$Z^{-1}\left[\frac{z^2}{(z-4)(z-3)}\right] = Z^{-1}\left[\frac{z}{z-4} \cdot \frac{z}{z-3}\right]$$

$$= Z^{-1}\left[\frac{z}{z-4}\right] * Z^{-1}\left[\frac{z}{z-3}\right]$$

W.K.T $Z[a^n] = \dfrac{z}{z-a}$

$\therefore \quad = (4)^n * (3)^n.$

$$f(n) * g(n) = \sum_{r=0}^{n} f(r)\, g(n-r).$$

$$= \sum_{r=0}^{n} (4)^{r} (3)^{n-r}.$$

$$= \sum_{r=0}^{n} 4^{r}\, 3^{n} \cdot 3^{-r}$$

$$= 3^{n} \sum_{r=0}^{n} 4^{r}\, 3^{-r} = 3^{n} \sum_{r=0}^{n} \left(\frac{4}{3}\right)^{r}$$

$$= 3^{n} \left[1 + \frac{4}{3} + \left(\frac{4}{3}\right)^{2} + \left(\frac{4}{3}\right)^{3} + \cdots + \left(\frac{4}{3}\right)^{n} \right]$$

Geometric Series $1 + a + a^2 + \cdots + a^{n-1} = \frac{a^n - 1}{a-1}$ (or) $\frac{1-a^n}{1-a}$

Sum to n terms $S_n = \frac{a^n - 1}{a-1}$ where n is the no. of terms

a is the variable /constant

$$Z^{-1}[\ \text{''}\ ] = 3^{n} \left[\frac{\left(\frac{4}{3}\right)^{n+1} - 1}{\frac{4}{3} - 1} \right]$$

$$= 3^{n} \left(\frac{3}{1}\right) \left[\frac{4^{n+1} - 3^{n+1}}{3^{n+1}} \right]$$

$$= \frac{3^{n+1} \left[4^{n+1} - 3^{n+1} \right]}{3^{n+1}}$$

$$\therefore Z^{-1}\left[\frac{z^2}{(z-4)(z-3)} \right] = (4)^{n+1} - (3)^{n+1}$$

2) $Z^{-1}\left[\frac{z^2}{(z-a)^2} \right]$ using Convolution theorem.

$$Z^{-1}\left[\frac{z^2}{(z-a)^2} \right] = Z^{-1}\left[\frac{z}{z-a} \cdot \frac{z}{z-a} \right]$$

$$= Z^{-1}\left[\frac{z}{z-a} \right] * Z^{-1}\left[\frac{z}{z-a} \right]$$

$$= (a)^{n} * (a)^{n}$$

$$= \sum_{r=0}^{n} (a)^{r} (a)^{n-r}$$

$$= \sum_{r=0}^{n} (a^r)(a)^{n-r}$$

$$= a^0 a^n + a^1 a^{n-1} + a^2 a^{n-2} + \cdots + a^n a^0$$

$$= a^n + a^n + a^n + a^n + \cdots + a^n \text{ (n+1 term)}$$

$$= (n+1) a^n$$

$$\therefore z^{-1}\left[\frac{z^2}{(z-a)^2}\right] = (n+1)a^n. //$$

3) Using Convolution Theorem

$$z^{-1}\left[\frac{z^2}{(z-1)(z-3)}\right]$$

4) find $z^{-1}\left[\frac{z^2}{(z+2)^2}\right]$ | 5) $z^{-1}\left[\frac{z^2}{(z+9)^2}\right]$

6) find $z^{-1}\left[\frac{z^2}{\left(z-\frac{1}{2}\right)\left(z-\frac{1}{4}\right)}\right]$.

7) find $z^{-1}\left[\frac{8z^2}{(2z-1)(4z+1)}\right]$

Soln: $z^{-1}\left[\frac{8z^2}{(2z-1)(4z+1)}\right] = z^{-1}\left[\frac{(8z^2/8)}{\left(\frac{2z-1}{2}\right)\left(\frac{4z+1}{4}\right)}\right]$

$$= z^{-1}\left[\frac{z^2}{\left(z-\frac{1}{2}\right)\left(z+\frac{1}{4}\right)}\right].$$

$$= z^{-1}\left[\frac{z}{\left(z-\frac{1}{2}\right)} \cdot \frac{z}{\left(z+\frac{1}{4}\right)}\right]$$

$$= Z^{-1}\left[\frac{z}{z-\frac{1}{2}}\right] * Z^{-1}\left[\frac{z}{z+\frac{1}{4}}\right]$$

$$= Z^{-1}\left[\frac{z}{z-\frac{1}{2}}\right] * Z^{-1}\left[\frac{z}{z-\left(-\frac{1}{4}\right)}\right]$$

formula: $Z^{-1}\left[\frac{z}{z-a}\right] = a^n$.

$$\therefore Z^{-1}[\ {}''\ ] = \left(\frac{1}{2}\right)^n * \left(\frac{-1}{4}\right)^n$$

$$= \left(\frac{-1}{4}\right)^n * \left(\frac{1}{2}\right)^n$$

$$= \sum_{r=0}^{n} \left(\frac{-1}{4}\right)^r \left(\frac{1}{2}\right)^{n-r}$$

$$= \sum_{r=0}^{n} \left(\frac{-1}{4}\right)^r \left(\frac{1}{2}\right)^n \left(\frac{1}{2}\right)^{-r}$$

$$= \left(\frac{1}{2}\right)^n \sum_{r=0}^{n} \left(\frac{-1}{4}\right)^r \left(\frac{1}{2}\right)^{-r}$$

$$= \left(\frac{1}{2}\right)^n \sum_{r=0}^{n} \left(\frac{-1}{4}\right)^r \left(\frac{1}{2^r}\right)^{-1}$$

$$= \left(\frac{1}{2}\right)^n \sum_{r=0}^{n} \left(\frac{-1}{4}\right)^r (2^r)$$

$$= \left(\frac{1}{2}\right)^n \sum_{r=0}^{n} \left(\frac{-2}{4}\right)^r$$

$$= \left(\frac{1}{2}\right)^n \sum_{r=0}^{n} \left(\frac{-1}{2}\right)^r$$

$$= \left(\frac{1}{2}\right)^n \left[\left(\frac{-1}{2}\right)^0 + \left(\frac{-1}{2}\right)^1 + \left(\frac{-1}{2}\right)^2 + \cdots\right]$$

$$=\left(\frac{1}{2}\right)^n\left[1+\left(\frac{-1}{2}\right)+\left(\frac{-1}{2}\right)^2+\cdots+\left(\frac{-1}{2}\right)^n\right]$$

Sum to n terms Here $a=\frac{-1}{2}$, $|a|<1$ (ii condition)

$$=\left(\frac{1}{2}\right)^n\left[\frac{1-\left(\frac{-1}{2}\right)^{n+1}}{1-\left(\frac{-1}{2}\right)}\right]$$

$$=\left(\frac{1}{2}\right)^n\left[\frac{1-\left(\frac{-1}{2}\right)^{n+1}}{1+\frac{1}{2}}\right]=\left(\frac{1}{2}\right)^n\left(\frac{2}{3}\right)\left(1-\left(\frac{-1}{2}\right)^{n+1}\right)$$

$$=\frac{2}{3}\left(\frac{1}{2}\right)^n\left[\left(1-\left(\frac{-1}{2}\right)^n\left(\frac{-1}{2}\right)\right)\right]$$

$$=\frac{2}{3}\left(\frac{1}{2}\right)^n\left[1+\frac{1}{2}\left(\frac{-1}{2}\right)^n\right]$$

$$=\left[\frac{2}{3}\left(\frac{1}{2}\right)^n+\frac{2}{3}\left(\frac{1}{2}\right)^n\left(\frac{1}{2}\right)\left(\frac{-1}{2}\right)^n\right]$$

$$=\frac{2}{3}\left(\frac{1}{2}\right)^n+\frac{1}{3}\left(\frac{1}{2}\times\frac{-1}{2}\right)^n$$

$$=\frac{2}{3}\left(\frac{1}{2}\right)^n+\frac{1}{3}\left(\frac{-1}{4}\right)^n$$

$$\therefore Z^{-1}\left[\frac{8z^2}{(2z-1)(4z+1)}\right]=\frac{2}{3}\left(\frac{1}{2}\right)^n+\frac{1}{3}\left(\frac{-1}{4}\right)^n$$