Solving of difference eqn using z-Transform

Eg. 1 Solve $y_{n+2} + 4y_{n+1} + 3y_n = 2^n$ with $y_0 = 0$ and $y_1 = 1$, using z-Transform.

Solution:

Taking z-Transform, we get

$$z[y_{n+2}] + 4z[y_{n+1}] + 3z[y_n] = z[2^n]$$

Results:

a) $z[y_n] = F(z)$

b) $z[y_{n+1}] = zF(z) - zy_0$

c) $z[y_{n+2}] = z^2F(z) - z^2y_0 - zy_1$

d) $z[y_{n+3}] = z^3F(z) - z^3y_0 - z^2y_1 - zy_2$

$$z^2F(z) - z^2y_0 - zy_1 + 4[zF(z) - zy_0] + 3F(z) = z[2^n]$$

Given: $y_0 = 0$ and $y_1 = 1$

$$z^2F(z) - 0 - z + 4zF(z) - 0 + 3F(z) = \frac{z}{z-2}$$

$$x^2F(x) - x + 4xF(x) + 3F(x) = \frac{x}{x-2}$$

$$x^2F(x) + 4xF(x) + 3F(x) = \frac{x}{x-2} + x$$

$$(x^2+4x+3)F(x) = \frac{x + x(x-2)}{x-2}$$

$$F(x) = \frac{x + x^2 - 2x}{(x-2)(x^2+4x+3)}$$

$$= \frac{x^2 - x}{(x-2)(x^2+4x+3)}$$

$$F(x) = \frac{x(x-1)}{(x-2)(x+1)(x+3)}$$

$$\frac{F(x)}{x} = \frac{x-1}{(x-2)(x+1)(x+3)}$$

Consider $\dfrac{x-1}{(x-2)(x+1)(x+3)} = \dfrac{A}{(x-2)} + \dfrac{B}{(x+1)} + \dfrac{C}{(x+3)}$

$$x-1 = A(x+1)(x+3) + B(x-2)(x+3) + C(x-2)(x+1)$$

put $x = -1$

$$-1 - 1 = 0 + B(-3)(2) + 0$$

$$-2 = -6B$$

$$\boxed{B = \frac{1}{3}}$$

put $x = 2$

$$1 = A(3)(5)$$

$$\boxed{A = \frac{1}{15}}$$

put $x = -3$

$$-4 = C(-5)(-2)$$

$$-4 = 10C$$

$$\boxed{C = \frac{-2}{5}}$$

$$\therefore \frac{x-1}{(x-2)(x+1)(x+3)} = \frac{1}{15(x-2)} + \frac{1}{3(x+1)} - \frac{2}{5(x+3)}$$

$$\therefore \frac{F(z)}{z} = \frac{1}{15(z-2)} + \frac{1}{3(z+1)} - \frac{2}{5(z+3)}$$

$$\Rightarrow F(z) = \frac{1}{15}\frac{z}{(z-2)} + \frac{1}{3}\frac{z}{(z+1)} - \frac{2}{5}\frac{z}{(z+3)}$$

Taking z^{-1} on both the sides

$$z^{-1}[F(z)] = \frac{1}{15} z^{-1}\left[\frac{z}{z-2}\right] + \frac{1}{3} z^{-1}\left[\frac{z}{z+1}\right] - \frac{2}{5} z^{-1}\left[\frac{z}{z+3}\right]$$

$$= \frac{1}{15}(2)^{n} + \frac{1}{3}(-1)^{n} - \frac{2}{5}(-3)^{n}, \quad n \geq 0.$$

since $z^{-1}\left[\frac{z}{z-a}\right] = (a)^{n}, \quad n \geq 0.$

$$f(n) = \frac{1}{15}(2)^{n} + \frac{1}{3}(-1)^{n} - \frac{2}{5}(-3)^{n} \;//$$

2) Solve the difference eqn using z-transform technique $y_{n+2} + 6y_{n+1} + 9y_{n} = 2^{n}$

Gn: $y_0 = y_1 = 0$

<u>Soln:</u> <u>Given:</u> $y_{n+2} + 6y_{n+1} + 9y_{n} = 2^{n}.$

Taking z-Transform, on both sides we get

$$z[y_{n+2}] + 6z[y_{n+1}] + 9z[y_n] = z[2^n]$$

$$\left[z^2F(z) - z^2y_0 - zy_1\right] + 6\left[zF(z) - zy_0\right] + 9F(z) = z[2^n]$$

$$z^2F(z) - z^2y_0 - zy_1 + 6zF(z) - 6zy_0 + 9F(z) = z[2^n]$$

$$z^2F(z) + 6zF(z) + 9F(z) - z^2y_0 - 6zy_0 - zy_1 = z[2^n]$$

$$[z^2+6z+9]F(z) + (z^2-6z)y_0 - zy_1 = \frac{z}{(z-2)}$$

$$(z^2+6z+9)F(z) = \frac{z}{z-2}$$

$$\frac{F(z)}{z} = \frac{1}{(z-2)(z^2+6z+9)}$$

Consider $\dfrac{1}{(z-2)(z^2+6z+9)} = \dfrac{1}{(z-2)(z+3)(z+3)}$

$$= \frac{1}{(z-2)(z+3)^2}$$

$$= \frac{A}{z-2} + \frac{B}{z+3} + \frac{C}{(z+3)^2}$$

$$1 = A(z+3)^2 + B(z+3)(z-2) + C(z-2)$$

put $z=2$

$A = \dfrac{1}{25}$

put $z=-3$

$C = \dfrac{-1}{5}$

put $z=0$

$1 = 9\left(\frac{1}{25}\right) + B(-6) + \frac{2}{5}$

$1 = \dfrac{9+10}{25} + (-6)B$

$\dfrac{25-19}{25} = (-6)B$

$\dfrac{6}{25} = -6B$

$\boxed{B = \dfrac{-1}{25}}$

$$\frac{F(z)}{z} = \frac{1}{25}\frac{1}{z-2} - \frac{1}{25}\frac{1}{z+3} - \frac{1}{5}\frac{1}{(z+3)^2}$$

$$\therefore F(z) = \frac{1}{25}\frac{z}{z-2} - \frac{1}{25}\frac{z}{z+3} - \frac{1}{5}\frac{z}{(z+3)^2}$$

formula.

$$Z^{-1}\left[\frac{z}{(z+a)^2}\right] = n(-a)^{n-1}$$

$$Z^{-1}[F(z)] = \frac{1}{25} Z^{-1}\left[\frac{z}{z-2}\right] - \frac{1}{25} Z^{-1}\left[\frac{z}{z+3}\right] - \frac{1}{5} Z^{-1}\left[\frac{z}{(z+3)^2}\right]$$

$$= \frac{1}{25}(2)^n - \frac{1}{25}(-3)^n - \frac{1}{5}\,2(-3)^{2-1}$$

$$f(n) = \frac{1}{25}2^n - \frac{1}{25}(-3)^n - \frac{2}{5}(-3)^1 \;//$$