



## UNIT 5 – Multiple Integrals

Double integration under a given region

Standard diagrams:

Equation	Graph.
1. $y = x$	
2. $x + y = 1$	
3. $x^2 + y^2 = a^2$	
4. $y^2 = 4ax$	
5. $x^2 = 4ay$	
6. $x = a, y = b$	



## UNIT 5 - Multiple Integrals

Double integration, Based on region

$\iint dx dy \rightarrow$  horizontal strip [left to right]  $\begin{matrix} x - \text{intervals of } y \\ y - \text{constant limit} \end{matrix}$

$\iint dy dx \rightarrow$  vertical strip [down to up]  $\begin{matrix} x \rightarrow \text{constant limit} \\ y \rightarrow \text{intervals of } x \end{matrix}$

1. Evaluate  $\iint_R xy \, dx dy$  where R is the 1<sup>st</sup> quadrant of the circle  $x^2 + y^2 = a^2$

Given :  $x^2 + y^2 = a^2$   
 $x^2 = a^2 - y^2$   
 $x = \pm \sqrt{a^2 - y^2}$

At y axis,  
 take  $x=0$ ,  $y^2 = a^2$   
 $y = \pm a$

Limits  $x : 0$  to  $\sqrt{a^2 - y^2}$   
 $y : 0$  to  $a$

Now,  $\iint_R xy \, dx dy = \int_0^a \int_0^{\sqrt{a^2 - y^2}} xy \, dx dy$

$= \int_0^a \left[ \frac{x^2 y}{2} \right]_0^{\sqrt{a^2 - y^2}} dy = \int_0^a \left[ \frac{(a^2 - y^2)y}{2} - 0 \right] dy$

$= \frac{1}{2} \int_0^a (a^2 y - y^3) dy = \frac{1}{2} \left[ \frac{a^2 y^2}{2} - \frac{y^4}{4} \right]_0^a$

$= \frac{1}{2} \left[ \left( \frac{a^4}{2} - \frac{a^4}{4} \right) - 0 \right] = \frac{a^4}{2} \left[ \frac{1}{2} - \frac{1}{4} \right]$

$= \frac{a^4}{2} \left( \frac{1}{4} \right)$

$\iint_R xy \, dx dy = \frac{a^4}{8}$



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2. Evaluate  $\iint_R xy(x+y) dy dx$  over the area between

$$y = x^2 \text{ and } y = x$$

Given:  $y = x^2$  and  $y = x$ .

$$\therefore x^2 = x$$

$$x^2 - x = 0 \Rightarrow x(x-1) = 0$$

$$x = 0 \text{ and } x = 1$$

When  $x = 0$ ,  $y = 0$

When  $x = 1$ ,  $y = 1$

Intersection point  $(0,0)$  &  $(1,1)$

Limits:  $x : 0 \text{ to } 1$

$y : x^2 \text{ to } x$

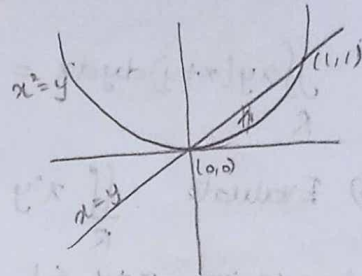
$$\begin{aligned} \iint_R xy(x+y) dy dx &= \int_0^1 \int_{x^2}^x xy(x+y) dy dx \\ &= \int_0^1 \int_{x^2}^x (x^2y + xy^2) dy dx \\ &= \int_0^1 \left[ \frac{x^2y^2}{2} + \frac{xy^3}{3} \right]_{y=x^2}^x dx \\ &= \int_0^1 \left[ \left( \frac{x^4}{2} + \frac{x^4}{3} \right) - \left( \frac{x^6}{2} + \frac{x^7}{3} \right) \right] dx \end{aligned}$$

$$= \int_0^1 \left( \frac{5x^4}{6} - \frac{x^6}{2} - \frac{x^7}{3} \right) dx$$

$$= \left[ \frac{5}{6} \cdot \frac{x^5}{5} - \frac{1}{2} \frac{x^7}{7} - \frac{1}{3} \frac{x^8}{8} \right]_0^1$$

$$= \left[ \frac{1}{6}(1) - \frac{1}{14}(1) - \frac{1}{24}(1) \right] = 0$$

$$= \frac{4-1}{24} - \frac{1}{14} = \frac{3}{24} - \frac{1}{14} = \frac{1}{8} - \frac{1}{14} = \frac{42-24}{336}$$



$dy dx \rightarrow$  Vertical strip  
 $y$  limit  $\rightarrow$  in terms of  $x$   
 $x$  limit  $\rightarrow$  constant limit



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$= \frac{18}{336}$

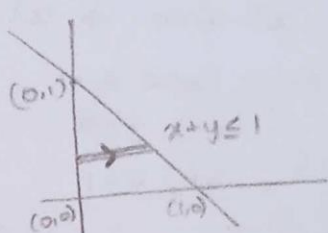
$$\iint_R xy(x+y) dy dx = \frac{3}{56}$$

3) Evaluate  $\iint_R x^2y dx dy$  over the region in the +ve quadrant for which  $x+y \leq 1$

Given  $x+y=1$   
 $y=0, x=1 \Rightarrow (1,0)$   
 $x=0, y=1 \Rightarrow (0,1)$

Intersection points  $(1,0)$  &  $(0,1)$

Limits:-  $x: 0$  to  $1-y$   
 $y: 0$  to  $1$



$dx dy \rightarrow$  horizontal strip  
 $x$  limit  $\rightarrow$  intervals of  $y$   
 $y$  limit  $\rightarrow$  constant limits

$$\begin{aligned} \iint_R x^2y dx dy &= \int_0^1 \int_0^{1-y} x^2y dx dy = \int_0^1 \left[ \frac{x^3}{3} y \right]_0^{1-y} dy \\ &= \int_0^1 \left[ \frac{(1-y)^3}{3} y - 0 \right] dy = \frac{1}{3} \int_0^1 y(1-3y+3y^2-y^3) dy \\ &= \frac{1}{3} \int_0^1 [y - 3y^2 + 3y^3 - y^4] dy \\ &= \frac{1}{3} \left[ \frac{y^2}{2} - \frac{3y^3}{3} + \frac{3y^4}{4} - \frac{y^5}{5} \right]_0^1 \\ &= \frac{1}{3} \left[ \frac{1}{2} - 1 + \frac{3}{4} - \frac{1}{5} \right] = \frac{1}{3} \left[ \frac{20 - 40 + 30 - 8}{40} \right] \\ &= \frac{1}{3} \left[ \frac{2}{40} \right] \end{aligned}$$

$$\iint_R x^2y dx dy = \frac{1}{60}$$



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4. Evaluate  $\iint_R x^2 y \, dy \, dx$  lies on the 1<sup>st</sup> quadrant of ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

Given that

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

$$\frac{y^2}{b^2} = 1 - \frac{x^2}{a^2} = \frac{a^2 - x^2}{a^2}$$

$$y^2 = b^2 \left( \frac{a^2 - x^2}{a^2} \right)$$

$$y = \pm \frac{b}{a} \sqrt{a^2 - x^2}$$

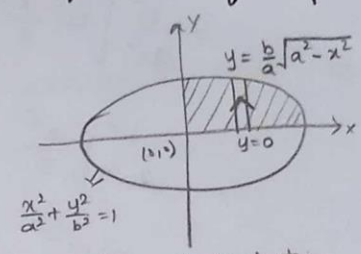
At x axis

$$y=0 \Rightarrow \frac{x^2}{a^2} + 0 = 1$$

$$x^2 = a^2$$

$$x = \pm a$$

Limits :  $x=0$  to  $a$   
 $y: 0$  to  $\frac{b}{a} \sqrt{a^2 - x^2}$



$dy \, dx \rightarrow$  Vertical strip  
 $y$  limit  $\rightarrow$  intervals of  $x$   
 $x$  limit  $\rightarrow$  constant limit

$$\iint_R x^2 y \, dy \, dx = \int_0^a \int_0^{\frac{b}{a} \sqrt{a^2 - x^2}} x^2 y \, dy \, dx$$

$$= \int_0^a x^2 \left[ \frac{y^2}{2} \right]_{y=0}^{\frac{b}{a} \sqrt{a^2 - x^2}} dx$$

$$= \int_0^a \frac{x^2}{2} \left[ \frac{b^2}{a^2} (a^2 - x^2) - 0 \right] dx$$

$$= \frac{b^2}{2a^2} \int_0^a x^2 (a^2 - x^2) dx$$

$$= \frac{b^2}{2a^2} \int_0^a (a^2 x^2 - x^4) dx$$



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$$\begin{aligned}
 &= \frac{b^2}{2a^2} \left[ a^2 \frac{x^3}{3} - \frac{x^5}{5} \right]_0^a \\
 &= \frac{b^2}{2a^2} \left[ \left( \frac{a^5}{3} - \frac{a^5}{5} \right) - 0 \right] \\
 &= \frac{b^2 a^5}{2a^2} \left[ \frac{1}{3} - \frac{1}{5} \right] \\
 &= \frac{b^2 a^3}{2} \left[ \frac{5-3}{15} \right] = \frac{b^2 a^3}{2} \left[ \frac{2}{15} \right] \\
 &= \frac{a^3 b^2}{15}
 \end{aligned}$$

$\iint_R x^2 y \, dy \, dx = \frac{a^3 b^2}{15}$

(i) Using double integration find the area enclosed by the curves  $y = 2x^2$  &  $y^2 = 4x$ .

$y = 2x^2 \rightarrow \textcircled{1}$   
 $y^2 = 4x \rightarrow \textcircled{2}$

Sub ① in ②  
 $(2x^2)^2 = 4x$   
 $4x^4 = 4x$   
 $x^4 - x = 0 \Rightarrow x(x^3 - 1) = 0$   
 $x = 0, x^3 = 1 \Rightarrow x = 1$

when  $x = 0, y = 0$   
 $x = 1, y = 2$

$x: 0 \text{ to } 1$   
 $y: 2x^2 \text{ to } 2\sqrt{x}$

$$\begin{aligned}
 A &= \int_0^1 \int_{2x^2}^{2\sqrt{x}} dy \, dx \\
 &= \int_0^1 [2\sqrt{x} - 2x^2] dx \\
 &= 2 \left[ \frac{x^{3/2}}{3/2} - \frac{2x^3}{3} \right]_0^1 \\
 &= \frac{2}{3}
 \end{aligned}$$