UNIT5–Multiple Integrals

Applications: Volume as triple integrals and solids of revolution

Problems based on volume.

1. find the volume of the sphere $x^2+y^2+z^2=a^2$ by triple integration

Given: $x^2+y^2+z^2=a^2$

$x^2 = a^2-y^2-z^2$

$x = \pm\sqrt{a^2-y^2-z^2}$

Limits

$x: 0$ to $\sqrt{a^2-y^2-z^2}$

$y: 0$ to $\sqrt{a^2-z^2}$

$z: 0$ to $a$

Take $x=0$,

$y^2+z^2=a^2$

$y^2=a^2-z^2$

$y=\pm\sqrt{a^2-z^2}$

Take $x=0, y=0$

$z^2=a^2$

$z=\pm a$

Volume of the sphere = $8\times$ Volume in the $1^{st}$ octant

$$= 8\iiint_V dx\,dy\,dz$$

$$= 8\int_0^a\int_0^{\sqrt{a^2-z^2}}\int_0^{\sqrt{a^2-y^2-z^2}} dx\,dy\,dz$$

$$= 8\int_0^a\int_0^{\sqrt{a^2-z^2}} [x]_0^{\sqrt{a^2-y^2-z^2}} dy\,dz$$

$$= 8\int_0^a\int_0^{\sqrt{a^2-z^2}} \left[\sqrt{a^2-y^2-z^2}-0\right] dy\,dz$$

$$= 8\int_0^a\int_0^{\sqrt{a^2-z^2}} \sqrt{(a^2-z^2)-y^2}\,dy\,dz$$

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$$= 8\int_0^a \int_0^{\sqrt{a^2-z^2}} \sqrt{\left(\sqrt{a^2-z^2}\right)^2 - y^2}\, dy\, dz$$

$$= 8\int_0^a \left[\frac{y\sqrt{a^2-z^2-y^2}}{2} + \frac{(a^2-z^2)}{2}\sin^{-1}\left(\frac{y}{\sqrt{a^2-z^2}}\right)\right]_0^{\sqrt{a^2-z^2}} dz$$

$$= 8\int_0^a \left\{\left[0 + \frac{a^2-z^2}{2}\sin^{-1}\left(\frac{\sqrt{a^2-z^2}}{\sqrt{a^2-z^2}}\right)\right] - (0+0)\right\} dz$$

$$= 8\int_0^a \frac{a^2-z^2}{2}\left(\frac{\pi}{2}\right) dz$$

$$= 2\pi \int_0^a (a^2-z^2)\, dz \qquad = 2\pi\left[a^2 z - \frac{z^3}{3}\right]_0^a$$

$$= 2\pi\left[\left(a^3 - \frac{a^3}{3}\right) - 0\right]$$

$$= 2\pi\left(\frac{2a^3}{3}\right)$$

$$= \frac{4\pi a^3}{3}$$

$\therefore$ Volume of the sphere $= \frac{4}{3}\pi a^3$

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2. Find the volume of the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$

Solution:-

Given $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$

$$\frac{x^2}{a^2}=1-\frac{y^2}{b^2}-\frac{z^2}{c^2}$$

$$x^2=a^2\left[1-\frac{y^2}{b^2}-\frac{z^2}{c^2}\right]$$

$$x=\pm a\sqrt{1-\frac{y^2}{b^2}-\frac{z^2}{c^2}}$$

Take $x=0$.

$$\frac{y^2}{b^2}=1-\frac{z^2}{c^2}$$

$$y^2=b^2\left(1-\frac{z^2}{c^2}\right)$$

$$y=\pm b\sqrt{1-\frac{z^2}{c^2}}$$

Take $x=0,\ y=0$.

$$\frac{z^2}{c^2}=1$$

$$z^2=c^2$$

$$z=\pm c$$

Limits :

$x$ : $0$ to $a\sqrt{1-\frac{y^2}{b^2}-\frac{z^2}{c^2}}$

$y$ : $0$ to $b\sqrt{1-\frac{z^2}{c^2}}$

$z$ : $0$ to $c$

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Volume = 8 × Volume in the $1^{st}$ quadrant

$$= 8 \iiint_R dx\,dy\,dz$$

$$= 8 \int_0^c \int_0^{b\sqrt{1-\frac{z^2}{c^2}}} \int_0^{a\sqrt{1-\frac{y^2}{b^2}-\frac{z^2}{c^2}}} dx\,dy\,dz$$

$$= 8 \int_0^c \int_0^{b\sqrt{1-\frac{z^2}{c^2}}} \left[ x \right]_0^{a\sqrt{1-\frac{y^2}{b^2}-\frac{z^2}{c^2}}} dy\,dz$$

$$= 8 \int_0^c \int_0^{b\sqrt{1-\frac{z^2}{c^2}}} a\sqrt{1-\frac{y^2}{b^2}-\frac{z^2}{c^2}}\; dy\,dz$$

$$= 8a \int_0^c \int_0^{b\sqrt{1-\frac{z^2}{c^2}}} \sqrt{\left(1-\frac{z^2}{c^2}\right)-\frac{y^2}{b^2}}\; dy\,dz$$

$$= 8a \int_0^c \int_0^{b\sqrt{1-\frac{z^2}{c^2}}} \sqrt{\frac{b^2\left(1-z^2/c^2\right)-y^2}{b^2}}\; dy\,dz$$

$$= \frac{8a}{b} \int_0^c \int_0^{b\sqrt{1-z^2/c^2}} \sqrt{\left(b\sqrt{1-z^2/c^2}\right)^2 - y^2}\; dy\,dz$$

$$= \frac{8a}{b} \int_0^c \left[ \frac{y}{2}\sqrt{b^2\left(1-\frac{z^2}{c^2}\right)-y^2} + \frac{b^2\left(1-z^2/c^2\right)}{2} \sin^{-1}\frac{y}{b\sqrt{1-\frac{z^2}{c^2}}} \right]_0^{b\sqrt{1-\frac{z^2}{c^2}}} dz$$

$$= \frac{8a}{b} \int_0^c \left[ \left(0 + \frac{b^2\left(1-z^2/c^2\right)}{2} \sin^{-1} \underbrace{\frac{b\sqrt{1-z^2/c^2}}{b\sqrt{1-z^2/c^2}}}_{1}\right) - 0 \right] dz$$

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$$= \frac{8a}{b}\int_0^c \frac{b^2}{2}\left(1-\frac{z^2}{c^2}\right)\sin^{-1}(1)\,dz$$

$$= 4ab\int_0^c \left(1-\frac{z^2}{c^2}\right)\frac{\pi}{2}\,dz$$

$$= 2\pi ab\int_0^c (1-z^2/c^2)\,dz$$

$$= 2\pi ab\left[z-\frac{z^3}{3c^2}\right]_0^c$$

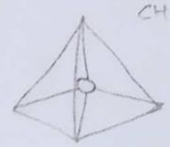
$$= 2\pi ab\left[c-\frac{c^3}{3c^2}\right]$$

$$= 2\pi ab\left[c-\frac{c}{3}\right]$$

$$= \frac{4}{3}\pi abc$$

$\therefore$ Volume of ellipsoid $= \frac{4}{3}\pi abc$.

3. Find the volume of the tetrahedron bound by the plane $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$




Given: $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$

$$\frac{x}{a} = 1-\frac{y}{b}-\frac{z}{c}$$

$$x = a\left(1-\frac{y}{b}-\frac{z}{c}\right)$$

Take $x=0$,

$$\frac{y}{b}+\frac{z}{c}=1$$

$$\frac{y}{b} = 1-z/c \qquad y = b\left(1-\frac{z}{c}\right)$$

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Take $x=0,\ y=0.$

$\frac{z}{c}=1$

$z=c$

Limits: $x : 0$ to $a\left(1-\frac{y}{b}-\frac{z}{c}\right)$

$y : 0$ to $b(1-z/c)$

$z : 0$ to $c$

$$\text{Volume} = \iiint_V dx\,dy\,dz$$

$$= \int_0^c \int_0^{b(1-\frac{z}{c})} \int_0^{a(1-\frac{y}{b}-\frac{z}{c})} dx\,dy\,dz$$

$$= \int_0^c \int_0^{b(1-\frac{z}{c})} \left[x\right]_0^{a[1-\frac{y}{b}-z/c]} dy\,dz$$

$$= \int_0^c \int_0^{b(1-\frac{z}{c})} \left[a\left(1-\frac{y}{b}-\frac{z}{c}\right)\right] dy\,dz$$

$$= a\int_0^c \int_0^{b(1-z/c)} \left[(1-z/c)-\frac{y}{b}\right] dy\,dz$$

$$= a\int_0^c \left[(1-z/c)y-\frac{y^2}{2b}\right]_0^{b(1-\frac{z}{c})} dz$$

$$= a\int_0^c \left[\left(1-\frac{z}{c}\right)b\left(1-\frac{z}{c}\right)-\frac{b^2(1-z/c)^2}{2b}\right] dz$$

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$$= a\int_0^c \left[\left(1-\frac{z}{c}\right)^2 b - \frac{b}{2}\left(1-z/c\right)^2\right]dz$$

$$= a\int_0^c \frac{b}{2}\left(1-\frac{z}{c}\right)^2 dz$$

$$= \frac{ab}{2}\int_0^c (1-z/c)^2 dz \qquad = \frac{ab}{2}\int_0^c \left(1+\frac{z^2}{c^2}-2\frac{z}{c}\right)dz$$

$$= \frac{ab}{2}\left[\frac{(1-z/c)^3}{(-1/c)3}\right]_0^c \qquad = \frac{ab}{2}\left[z+\frac{z^3}{3c^2}-\frac{\cancel{2}z^2}{\cancel{2}c}\right]_0^c$$

$$= \frac{-abc}{6}[0-1] \qquad = \frac{ab}{2}\left[c+\frac{c^3}{3c^2}-\frac{c^2}{c}-0\right]$$

$$= \frac{abc}{6} \qquad = \frac{ab}{2}\left[\cancel{c}+\frac{c}{3}-\cancel{c}\right]$$

$$= \frac{abc}{6}$$

$$\therefore \text{Volume} = \frac{abc}{6}.$$