3. Solution of Equations

Newton Method (or) Newton Raphson Method

Netwon Raphson method is extensively used for analysis of flow in water distribution networks. It is used to find the roots of non linear equations.

Formula:

$$x_{n+1} = x_n - \frac{F(x_n)}{F'(x_n)}, \text{ Provided } F'(x_n) \neq 0.$$

order = 2.

Newton Raphson condition:

$$|F(x)F''(x)| < |F'(x)|^2.$$

Problems:

1. Find the smallest positive root of the eqn

$x^3 - 2x + 0.5 = 0.$

Let $F(x) = x^3 - 2x + 0.5$

Now $F'(x) = 3x^2 - 2$

$F(0) = 0.5$

$F(1) = -0.5$

$\therefore$ The ~~pa~~ root lies b/w 0 & 1

Since $|F(0)| = |F(1)|$

Let us assume $x_0 = 0$..

Newton Raphson Formula :-

$$x_{n+1} = x_n - \frac{F(x_n)}{F'(x_n)}$$

Put $n=0$, $x_1 = x_0 - \frac{F(x_0)}{F'(x_0)} = 0 - \frac{0.5}{-2} = 0.25$

Put $n=1$, $x_2 = x_1 - \frac{F(x_1)}{F'(x_1)} = 0.25 - \frac{F(0.25)}{F'(0.25)}$

$$x_2 = 0.2586$$

Put $n=2$, $x_3 = x_2 - \frac{F(x_2)}{F'(x_2)}$

$$= 0.2586.$$

Since x_2 & x_3 are equal roots, the smallest positive root is 0.2586.

d) Compute the real root of $x \log x = 1.2$ correct to 3 decimal places using Newton Raphson method.

Let $f(x) = x \log x - 1.2$

$$F'(x) = x\left(\frac{1}{x}\right) + \log x (1) - 0 = 1 + \log x$$

Now $F(0) = -ve$

$F(1) = -1.2$

$F(2) = -0.5979$ $\therefore$ The root lies between

$F(3) = 0.2314$ 2 & 3

Since $|F(2)| > |F(3)|$. Let us assume $x_0 = 3$

Newton - Raphson formula,

$$x_{n+1} = x_n - \frac{F(x_n)}{F'(x_n)}$$

$$x_1 = x_0 - \frac{F(x_0)}{F'(x_0)} = 2.8434$$

$$x_2 = 2.8434 - \frac{F(2.8434)}{F'(2.8434)}$$

$$= 2.7822$$

$$x_3 = 2.7822 - \frac{F(2.7822)}{F'(2.7822)}$$

$$= 2.7576$$

lly,
$x_4 = 2.7476$

$x_5 = 2.7435$

$x_6 = 2.7418$

$x_7 = 2.7411$

$x_8 = 2.7408$

$x_9 = 2.7407$

$x_{10} = 2.7407$

$\therefore$ The required root is 2.7407

HW find the +ve root of $2x^3 - 3x - 6 = 0$

Ans: 1.7838

8. Find the -ve root of $x^3 - \sin x + 1 = 0$.

[radian mode]

Let $F(x) = x^3 - \sin x + 1$

$F'(x) = 3x^2 - \cos x$

$F(0) = 1$

$F(-1) = 0.8415$ (+ve)

$F(-2) = -6.0907$ (-ve)

$\therefore$ The root lies between -1 & -2

Since $|F(-1)| < |F(-2)|$. Let us assume $x_0 = -1$

$$x_1 = x_0 - \frac{F(x_0)}{F'(x_0)}$$

$x_1 = \cancel{-1.3421}$ -1.3421

$$x_2 = -1.3421 - \frac{F(-1.3421)}{F'(-1.3421)}$$

$= -1.2564$

$x_3 = -1.2491$

$x_4 = -1.2491$

Since x_3 & x_4 are equal

$\therefore$ The required -ve root is -1.2491

(1) obtain Newton's Iterative formula for finding $\sqrt{N}$ where N is a +ve real no. Hence evaluate $\sqrt{5}$

Let $x = \sqrt{N} \Rightarrow x^2 = N \Rightarrow x^2 - N = 0.$

$F(x) = x^2 - N$

$F'(x) = 2x$

Now $x_{n+1} = x_n - \frac{F(x_n)}{F'(x_n)}$

$= x_n - \frac{x_n^2 - N}{2x_n} = \frac{2x_n^2 - x_n^2 + N}{2x_n}$

$= \frac{x_n^2 + N}{2x_n}$, which is an iterative formula for $\sqrt{N}$

To find $\sqrt{5}$

$x = \sqrt{5}$

$x^2 - 5 = 0$

$F(x) = x^2 - 5$; $F'(x) = 2x$.

$F(0) = -5$ (-ve)

$F(1) = 1 - 5 = -4$

$F(2) = 4 - 5 = -1$ (-ve)

$F(3) = 9 - 5 = 4$ (+ve)

$\therefore$ The root lies between 2 & 3

$\because |F(2)| < |F(3)|$, let us assume that $x_0 = 2$

Now, $x_{n+1} = \frac{x_n^2 + N}{2x_n} \Rightarrow n=0 \; x_1 = \frac{x_0^2+5}{2x_0} = \frac{4+5}{2(2)} = \frac{9}{4}$

$$x_1 = 2.25$$

$$x_2 = \frac{x_1^2 + N}{2x_1} = \frac{(2.25)^2+5}{2(2.25)}$$

$$= 2.2361$$

$$x_3 = \frac{(2.2361)^2+5}{2(2.2361)} = 2.2361$$

$\therefore$ The value of $\sqrt{5} = 2.2361$.

2) Find the iterative formula for finding the value of $\frac{1}{N}$, where N is a real no using Newton Raphson method. Hence evaluate $\frac{1}{26}$ correct to 4 decimal places.

Let $x = \frac{1}{N}$ (i.e.) $N = \frac{1}{x}$

$F(x) = \frac{1}{x} - N$; $F'(x) = \frac{-1}{x^2}$

Now, $x_{n+1} = x_n - \frac{F(x_n)}{F'(x_n)} = x_n - \frac{\frac{1}{x_n} - N}{-1/x_n^2}$

$= x_n + x_n(1 - Nx_n) = x_n + x_n - Nx_n^2$

$x_{n+1} = 2x_n - Nx_n^2$, which is the iterative formula.

To find $\frac{1}{26}$, $N=26$

$$F(x)=\frac{1}{x}-26 \; ; \; F'(x)=\frac{-1}{x^2}$$

$F(0) = -26$ (-ve)

$F(1) = -25$

$F(2) = -25.5$ (-ve)

Let us take $x_0 = \frac{1}{25} = 0.04$, nearer to given N

Let $x_0 = 0.04$

wkt $x_{n+1} = 2x_n - Nx_n^2$

$$x_1 = 2(0.04) - 26(0.04)^2$$

$$x_1 = 0.0384$$

$$x_2 = 0.0384$$

Since x_1 & x_2 are equal, the value of $\frac{1}{26} = 0.0384$

3) Derive Newton's algorithm for finding the p^{th} root of a number N & find the value of $(24)^{1/3}$

Let $x = N^{1/p}$

$$x^p = N$$

$$\Rightarrow x^p - N = 0$$

Let $F(x) = x^p - N$; $F'(x) = px^{p-1}$

wkt, $$x_{n+1} = x_n - \frac{F(x_n)}{F'(x_n)} = x_n - \frac{x_n^p - N}{px_n^{p-1}}$$

$$x_{n+1} = \frac{Px_n^P - x_n^P + N}{Px_n^{P-1}} = \frac{(P-1)x_n^P + N}{Px_n^{P-1}}$$

To find $(24)^{1/3}$

Here $N = 24$, $P = 3$.

$F(x) = x^P - N$

$F(x) = x^3 - 24$

$F(0) = -24$

$F(1) = -23$

$F(2) = -16$ (-ve)

$F(3) = 3$ (+ve), the root lies between 2 & 3

Since $|F(2)| > |F(3)|$ let us assume $x_0 = 3$

$$x_{n+1} = \frac{(3-1)x_n^3 + 24}{3x_n^{3-1}} = \frac{2x_n^3 + 24}{3x_n^2}$$

$$x_1 = \frac{2x_0^3 + 24}{3x_0^2} = 2.8888$$

$x_2 = 2.8845$

$x_3 = 2.8844$

$x_4 = 2.8844$

Since $x_3 = x_4$, the required root is 2.8844

Gauss Jordan Method

In Gauss Jordan method, the coefficient matrix is reduced to a diagonal matrix (or even a unit matrix) rather than a triangular matrix as in the Gaussian method. Here the elimination of the unknowns is done not only in the equation below. but also in the eqns above the leading diagonal. Here we get the solution without using the back substitution method.

1. Solve by Gauss Jordan method

$10x+y+z=12$

$2x+10y+z=13$

$x+y+5z=7$

Augmented matrix form

$$\begin{pmatrix} 10 & 1 & 1 & 12 \\ 2 & 10 & 1 & 13 \\ 1 & 1 & 5 & 7 \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & 1/10 & \frac{1}{10} & \frac{12}{10} \\ 2 & 10 & 1 & 13 \\ 1 & 1 & 5 & 7 \end{pmatrix} R_1 \to \frac{R_1}{10}$$

$$\sim \begin{pmatrix} 1 & \frac{1}{10} & \frac{1}{10} & \frac{12}{10} \\ 0 & \frac{49}{5} & \frac{4}{5} & \frac{53}{5} \\ 0 & \frac{9}{10} & \frac{49}{10} & \frac{58}{10} \end{pmatrix} \begin{matrix} R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - R_1 \end{matrix}$$

$$\sim \begin{pmatrix} 1 & \frac{1}{10} & \frac{1}{10} & \frac{12}{10} \\ 0 & 1 & \frac{4}{49} & \frac{53}{49} \\ 0 & \frac{9}{10} & \frac{49}{10} & \frac{58}{10} \end{pmatrix} R_2 \to R_2 \div \frac{49}{5}$$

$$\sim \begin{pmatrix} 1 & \frac{1}{10} & \frac{1}{10} & \frac{12}{10} \\ 0 & 1 & \frac{4}{49} & \frac{53}{49} \\ 0 & 0 & 4.8266 & 4.8266 \end{pmatrix} R_3 \to R_3 - \frac{9}{10} R_2$$

$$\sim \begin{pmatrix} 1 & \frac{1}{10} & \frac{1}{10} & \frac{12}{10} \\ 0 & 1 & \frac{4}{49} & \frac{53}{49} \\ 0 & 0 & 1 & 1 \end{pmatrix} R_3 \to \frac{R_3}{4.8266}$$

$$\sim \begin{pmatrix} 1 & \frac{1}{10} & 0 & \frac{11}{10} \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix} \begin{matrix} R_1 \to R_1 - \frac{1}{10} R_3 \\ R_2 \to R_2 - \frac{4}{49} R_3 \end{matrix}$$

$$\sim \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix} R_1 \to R_1 - \frac{1}{10} R_2$$

$$\therefore \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$$\Rightarrow x=1,\ y=1,\ z=1$$

Procedure:-

1. Write the augmented matrix

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & b_1 \\ a_{21} & a_{22} & a_{23} & b_2 \\ a_{31} & a_{32} & a_{33} & b_3 \end{pmatrix}$$ for the given system of equations

2. Using elementary row transformation reduce the given matrix into a diagonal matrix say

$$\begin{pmatrix} c_{11} & 0 & 0 & d_1 \\ 0 & c_{22} & 0 & d_2 \\ 0 & 0 & c_{33} & d_3 \end{pmatrix}$$

3) From the above matrix we can find the values of x, y and z.

Q. Solve $x+3y+3z=16$, $x+4y+3z=18$, $x+3y+4z=19$ by Gauss-Jordan Method

Given:

$$x+3y+3z=16$$
$$x+4y+3z=18$$
$$x+3y+4z=19$$

The Augmented matrix is

$$\begin{pmatrix} 1 & 3 & 3 & 16 \\ 1 & 4 & 3 & 18 \\ 1 & 3 & 4 & 19 \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & 3 & 3 & 16 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{pmatrix} \begin{matrix} R_2 \to R_2 - R_1 \\ R_3 \to R_3 - R_1 \end{matrix}$$

$$\begin{pmatrix} 1 & 0 & 3 & 10 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{pmatrix} \quad R_1 \to R_1 - 3R_2$$

$$\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{pmatrix} \quad R_1 \to R_1 - 3R_3.$$

The matrix finally reduces to the form given by

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

$\therefore$ $x = 1$, $y = 2$, $z = 3$.

Iterative methods:

i) Gauss-Jacobi method

ii) Gauss Seidel method

Gauss Seidal Method:-

Let the system of simultaneous equation be

$$\left.\begin{aligned} a_{11}x_1 + b_{12}y + c_{13}z &= d_1 \\ a_{21}x + b_{22}y + c_{23}z &= d_2 \\ a_{31}x + b_{32}y + c_{33}z &= d_3 \end{aligned}\right\} \to (1)$$

The above system can be written as

$$x_1 = \frac{1}{a_{11}}\left(d_1 - a_{12}\cancel{y}x_2 - a_{13}\cancel{z}x_3\right) \to (2)$$

$$y_1 = \frac{1}{a_{22}}\left(d_2 - a_{21}x_1 - a_{23}x_3\right) \to (3)$$

$$x_3 = \frac{1}{a_{33}}(d_3 - a_{31}x - a_{32}y) \rightarrow (4)$$

3) Substituting ~~y~~ $= 0$ and $z = 0$ in (1), we get the value for x and it is denoted by x_1:

Substituting $x = x_1$ and $z = 0$ we get the value for ~~x~~ y. and it is denoted by y_1. Substituting $x = x_1$ and $y = y_1$ in 3 we get the value for z and it is denoted by z_1. These values of x_1, y_1, z_1 are called first iterative values of x, y and z.

4. The above process may continue for second, third, fourth, etc. iterations.

1) Solve by Gauss Seidel method

$$x + y + 54z = 110$$
$$27x + 6y - 5z = 85$$
$$6x + 15y + 2z = 72$$

Let us rearrange the equations

$$27x + 6y - 5z = 85 \rightarrow (1)$$
$$6x + 15y + 2z = 72 \rightarrow (2)$$
$$x + y + 54z = 110 \rightarrow (3)$$

$$\therefore |27| > |6| + |5|$$
$$|15| > |6| + |2|$$
$$|54| > |1| + |1|$$

① $\Rightarrow$ $x = \dfrac{85 - 6y + 5z}{27}$

② $\Rightarrow$ $y = \dfrac{72 - 6x - 2z}{15}$

③ $\Rightarrow$ $z = \dfrac{110 - x - y}{54}$

Let $y_0 = z_0 = 0.$

$x = \frac{1}{27}[85 - 6y + 5z]$	$y = \frac{1}{15}[72 - 6x - 2z]$	$z = \frac{1}{54}[110 - x - y]$
$x_1 = 3.148$	$y_1 = 3.5408$	$z_1 = 1.913$
$x_2 = 2.715$	$y_2 = 3.458$	$z_2 = 1.923$
$x_3 = 2.735$	$y_3 = 3.449$	$z_3 = 1.923$
$x_4 = 2.738$	$y_4 = 3.4484$	$z_4 = 1.9224$
$x_5 = 2.738$	$y_5 = 3.4484$	$z_5 = 1.9224$

$\therefore \quad x = 2.738$

$y = 3.4484$

$z = 1.9224$

2. Solve the following System by Gauss Seidal method: $9x - y + 2z = 9$

$x + 10y - 2z = 15$

$2x - 2y - 13z = -17$

The given System of eqns are

$9x - y + 2z = 9 \rightarrow (1)$

$x + 10y - 2z = 15 \rightarrow (2)$

$2x - 2y - 13z = -17 \rightarrow (3)$.

Clearly the coeff matrix is diagonally dominant,

So we can apply Gauss seidal method

$x = \frac{1}{9}[9 + y - 2z]$

$y = \frac{1}{10}[15 - x + 2z]$

$z = \frac{+1}{13}[17 + 2x - 2y]$

Let $y_0 = z_0 = 0$.

$x = \frac{1}{9}[9 + y - 2z]$	$y = \frac{1}{10}[15 - x + 2z]$	$z = \frac{1}{13}[17 + 2x - 2y]$
$x_1 = 1$	$y_1 = 1.4$	$z_1 = 1.246$
$x_2 = 0.8786$	$y_2 = 1.6613$	$z_2 = 1.1872$
$x_3 = 0.9208$	$y_3 = 1.6454$	$z_3 = 1.1962$

$x_4 = 0.9170$	$y_4 = 1.6475$	$z_4 = 1.1953$
$x_5 = 0.9174$	$y_5 = 1.6473$	$z_5 = 1.1954$
$x_6 = 0.9174$	$y_6 = 1.6473$	$z_6 = 1.1954$

$\therefore x = 0.9174, \; y = 1.6473, \; z = 1.1954$

Gauss - Jacobi Method

Let the system of simultaneous equations be

$$\left.\begin{aligned} a_1x + b_1y + c_1z &= d_1 \\ a_2x + b_2y + c_2z &= d_2 \\ a_3x + b_3y + c_3z &= d_3 \end{aligned}\right\} \rightarrow (1)$$

Assume, $|a_1| > |b_1| + |c_1|$

$|b_2| > |a_2| + |c_2|$

$|c_3| > |a_3| + |c_3|$

ie. The diagonal elements should dominant, so that the iteration process can.

This system of equations can also be written as

$$\left.\begin{aligned} x &= \frac{1}{a_1}(d_1 - b_1y - c_1z) \\ y &= \frac{1}{b_2}(d_2 - a_2x - c_2z) \\ z &= \frac{1}{c_3}(d_3 - a_3x - b_3y) \end{aligned}\right\} \rightarrow (2)$$

Solve the following system by Gauss Jacobi method

$10x - 5y - 2z = 3$

$4x - 10y + 3z = -3$

$x + 6y + 10z = -3$

Soln:- $\because$ $|10| > |-5| + |-2|$

$|-10| > |4| + |3|$

$|10| > |1| + |6|$

Since the diagonal elements are dominant, the iteration process can be applied.

The given system can be written as,

$$x = \frac{1}{10}(3 + 5y + 2z)$$

$$y = \frac{1}{-10}(3 + 4x + 3z)$$

$$z = \frac{1}{10}(-3 - x - 6y)$$

$x = \frac{1}{10}(3+5y+2z)$	$y = \frac{-1}{10}(3+4x+3z)$	$z = \frac{1}{10}(-3-x-6y)$
$x_1 = 0.3$	$y_1 = 0.3$	$z_1 = -0.3$
$x_2 = 0.39$	$y_2 = 0.33$	$z_2 = -0.51$
$x_3 = 0.363$	$y_3 = 0.303$	$z_3 = -0.537$
$x_4 = 0.3441$	$y_4 = 0.2841$	$z_4 = -0.5181$
$x_5 = 0.3384$	$y_5 = 0.2822$	$z_5 = -0.5048$
$x_6 = 0.3401$	$y_6 = 0.2829$	$z_6 = -0.5031$
$x_7 = 0.3413$	$y_7 = 0.2751$	$z_7 = -0.5043$
$x_8 = 0.3416$	$y_8 = 0.2852$	$z_8 = -0.5051$
$x_9 = 0.3415$	$y_9 = 0.28511$	$z_9 = -0.5052$
$x_{10} = 0.34148$	$y_{10} = 0.28504$	$z_{10} = -0.5052$

$x \simeq 0.3415$, $y \simeq 0.2850$ $z \simeq -0.5052$

(2) Solve the following using gauss Jacobi's iteration method: $30x - 2y + 3z = 75$

$$x + 17y - 2z = 48$$

$$x + y + 9z = 15$$

Soln: $a_1 : |30| > |-2| + |3|$

$b_2 : |17| > |1| + |-2|$

$c_3 : |9| > |1| + |1|$

Since the diagonal elements are dominant. The iteration process is applied here:

The given system can be written as:

$$x = \frac{1}{30}[75 + 2y - 3z]$$

$$y = \frac{1}{17}[48 - x + 2z]$$

$$z = \frac{1}{9}[15 - x - y]$$

$x = \frac{1}{30}[75+2y-3z]$	$y = \frac{1}{17}[48-x+2z]$	$z = \frac{1}{9}[15-x-y]$
$x_1 = 2.5$	$y_1 = 2.8235$	$z_1 = 1.6667$
$x_2 = 2.5217$	$y_2 = 2.8725$	$z_2 = 1.0757$
$x_3 = 2.5839$	$y_3 = 2.8016$	$z_3 = 1.0673$
$x_4 = 2.5800$	$y_4 = 2.7971$	$z_4 = 1.0682$
$x_5 = 2.5796$	$y_5 = 2.7974$	$z_5 = 1.0692$
$x_6 = 2.5795$	$y_6 = 2.7975$	$z_6 = 1.0692$
$x_7 = 2.5795$	$y_7 = 2.7975$	$z_7 = 1.0692$

$\therefore$ $x = 2.5795$, $y = 2.7975$, $z = 1.0692$

Gauss Elimination Method :

1. Write the augmented matrix $\begin{pmatrix} a_{11} & a_{12} & a_{13} & b_1 \\ a_{21} & a_{22} & a_{23} & b_2 \\ a_{31} & a_{32} & a_{33} & b_3 \end{pmatrix}$ for the given system of simultaneous equation.

2. Using elementary row operations reduce the given matrix into an upper-triangular matrix say

$$\begin{pmatrix} c_{11} & c_{12} & c_{13} & d_1 \\ 0 & c_{22} & c_{23} & d_2 \\ 0 & 0 & c_{33} & d_3 \end{pmatrix}$$

3. By Back substitution we get the values for x, y & z.

1. Solve the system of equations

$$28x + 4y - z = 32$$
$$x + 3y + 10z = 24$$
$$2x + 17y + 4z = 35$$

by using Gauss elimination method.

The augmented matrix is

$$\begin{pmatrix} 28 & 4 & -1 & 32 \\ 1 & 3 & 10 & 24 \\ 2 & 17 & 4 & 35 \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & \frac{1}{7} & \frac{-1}{28} & \frac{8}{7} \\ 1 & 3 & 10 & 24 \\ 2 & 17 & 4 & 35 \end{pmatrix} R_1 \to R_1 \div 28$$

$$\sim \begin{pmatrix} 1 & \frac{1}{7} & \frac{-1}{28} & \frac{8}{7} \\ 0 & \frac{20}{7} & \frac{281}{28} & \frac{160}{7} \\ 0 & \frac{117}{7} & \frac{57}{14} & \frac{229}{7} \end{pmatrix} \begin{array}{l} R_2 \to R_2 - R_1 \\ R_3 \to R_3 - 2R_1 \end{array}$$

$$\sim \begin{pmatrix} 1 & \frac{1}{7} & \frac{-1}{28} & \frac{8}{7} \\ 0 & \frac{117}{7} & \frac{57}{14} & \frac{229}{7} \\ 0 & \frac{20}{7} & \frac{281}{28} & \frac{160}{7} \end{pmatrix} \quad R_2 \leftrightarrow R_3$$

$$\sim \begin{pmatrix} 1 & \frac{1}{7} & \frac{-1}{28} & \frac{8}{7} \\ 0 & 1 & \frac{19}{78} & \frac{229}{117} \\ 0 & 0 & \frac{1457}{156} & \frac{2020}{117} \end{pmatrix} \quad \begin{matrix} R_2 \to R_2 \times \frac{7}{117} \\ R_3 \to R_3 - \frac{20}{7} R_2 \end{matrix}$$

$$\sim \begin{pmatrix} 1 & \frac{1}{7} & \frac{-1}{28} & \frac{8}{7} \\ 0 & 1 & \frac{19}{78} & \frac{229}{117} \\ 0 & 0 & 1 & 1.8485 \end{pmatrix} \quad R_3 \to R_3 \times \frac{156}{1457}$$

Using back substitution,

$z = 1.8485$

$y + \frac{19}{78} z = \frac{229}{117}$

$y = \frac{229}{117} - \frac{19}{78}(1.8485)$

$= 1.50697$

$x + \frac{1}{7} y - \frac{1}{28} z = \frac{8}{7}$

$x = \frac{8}{7} + \frac{1}{28}(1.8485) - \frac{1}{7}(1.5069)$

$= 0.9936$

$\therefore x = 0.9936, \ y = 1.50697, \ z = 1.8485$

2) Solve the system of equations by gauss elimination method.

$$10x - 2y + 3z = 23$$
$$2x + 10y - 5z = -33$$
$$3x - 4y + 10z = 41$$

The augmented matrix is

$$\begin{pmatrix} 10 & -2 & 3 & 23 \\ 2 & 10 & -5 & -33 \\ 3 & -4 & 10 & 41 \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & \frac{-1}{5} & \frac{3}{10} & \frac{23}{10} \\ 0 & \frac{52}{5} & \frac{-28}{5} & \frac{-188}{5} \\ 0 & \frac{-17}{5} & \frac{91}{10} & \frac{341}{10} \end{pmatrix} \begin{matrix} R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - 3R_1 \end{matrix}$$

$$\sim \begin{pmatrix} 1 & \frac{-1}{5} & \frac{3}{10} & \frac{23}{10} \\ 0 & 1 & \frac{-7}{13} & \frac{-47}{13} \\ 0 & \frac{-17}{5} & \frac{91}{10} & \frac{341}{10} \end{pmatrix} R_2 \to R_2 \div \frac{52}{5}$$

$$\sim \begin{pmatrix} 1 & \frac{-1}{5} & \frac{3}{10} & \frac{23}{10} \\ 0 & 1 & \frac{-7}{13} & \frac{-47}{13} \\ 0 & 0 & \frac{189}{26} & \frac{567}{26} \end{pmatrix} R_3 \to R_3 + \left(\frac{17}{5}\right) R_2.$$

$$\sim \begin{pmatrix} 1 & \frac{-1}{5} & \frac{3}{10} & \frac{23}{10} \\ 0 & 1 & \frac{-7}{13} & \frac{-47}{13} \\ 0 & 0 & 1 & 3 \end{pmatrix} R_3 \to R_3 \div \frac{189}{26}$$

By using Back substitution,

$$z = 3$$

$$y - \frac{7}{13} z = \frac{-47}{13} \Rightarrow y = \frac{-47}{13} + \frac{7}{13}(3)$$

$$y = \frac{-47+21}{13} = \frac{-26}{13} = -2$$

$$\therefore y = -2$$

$$\Rightarrow x - \frac{1}{5}y + \frac{3}{10}z = \frac{23}{10}.$$

$$x = \frac{23}{10} + \frac{1}{5}y \mp \frac{3}{10}z$$

$$= \frac{23}{10} + \frac{1}{5}(-2) \mp \frac{3}{10}(3)$$

$$= \frac{23}{10} - \frac{2}{5} \mp \frac{9}{10} = \frac{23-4\mp 9}{10}$$

$$= \frac{23-13}{10} = \frac{10}{10} = 1$$

$$\therefore x = 1,\ y = -2,\ z = 3.$$

Inverse Gauss Jordan Method

1. Using Gauss Jordan method find the inverse of the matrix $\begin{pmatrix} 2 & 2 & 3 \\ 2 & 1 & 1 \\ 1 & 3 & 5 \end{pmatrix}$

Let $A = \begin{pmatrix} 2 & 2 & 3 \\ 2 & 1 & 1 \\ 1 & 3 & 5 \end{pmatrix}$, $X = \begin{pmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{pmatrix}$ be the inverse of A

So that $AX = I$

The augmented matrix is,

$$\left[\begin{array}{cccccc} 2 & 2 & 3 & 1 & 0 & 0 \\ 2 & 1 & 1 & 0 & 1 & 0 \\ 1 & 3 & 5 & 0 & 0 & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{cccccc} 2 & 2 & 3 & 1 & 0 & 0 \\ 1 & -1 & -2 & -1 & 1 & 0 \\ 0 & 4 & 7 & -1 & 0 & 2 \end{array}\right] \begin{array}{l} R_2 \to R_2 - R_1 \\ R_3 \to 2R_3 - R_1 \end{array}$$

$$\sim \left[\begin{array}{cccccc} 2 & 2 & 3 & 1 & 0 & 0 \\ 0 & -1 & -2 & -1 & 1 & 0 \\ 0 & 0 & -1 & -5 & 4 & 2 \end{array}\right] R_3 \to R_3 + 4R_2$$

$$\sim \left[\begin{array}{cccccc} 2 & 2 & 3 & 1 & 0 & 0 \\ 0 & -1 & 0 & 9 & -7 & -4 \\ 0 & 0 & -1 & -5 & 4 & 2 \end{array}\right] R_2 \to R_2 - 2R_3$$

$$\sim \begin{bmatrix} 2 & 0 & 3 & 19 & -14 & -8 \\ 0 & -1 & 0 & 9 & -7 & -4 \\ 0 & 0 & -1 & -5 & 4 & 2 \end{bmatrix} R_1 \to R_1 + 2R_2$$

$$\sim \begin{bmatrix} 2 & 0 & 0 & 4 & -2 & -2 \\ 0 & -1 & 0 & 9 & -7 & -4 \\ 0 & 0 & -1 & -5 & 4 & 2 \end{bmatrix} R_1 \to R_1 + 3R_3$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 2 & -1 & -1 \\ 0 & 1 & 0 & -9 & 7 & 4 \\ 0 & 0 & 1 & 5 & -4 & -2 \end{bmatrix} \begin{matrix} R_1 \to \frac{R_1}{2} \\ R_2 \to (-1) \times R_2 \\ R_3 \to (-1) \times R_3. \end{matrix}$$

Hence the inverse of the given matrix A is

$$\begin{bmatrix} 2 & -1 & -1 \\ -9 & 7 & 4 \\ 5 & -4 & -2 \end{bmatrix}$$

2) Find the inverse of $\begin{pmatrix} 2 & 2 & 6 \\ 2 & 6 & -6 \\ 4 & -8 & -8 \end{pmatrix}$ using gauss Jordan method.

Let $A = \begin{pmatrix} 2 & 2 & 6 \\ 2 & 6 & -6 \\ 4 & -8 & -8 \end{pmatrix}$ and

$X = \begin{pmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{pmatrix}$ be the inverse of A

so that $AX = I$

The Augmented matrix is

$$\left[\begin{array}{cccccc} 2 & 2 & 6 & 1 & 0 & 0 \\ 2 & 6 & -6 & 0 & 1 & 0 \\ 4 & -8 & -8 & 0 & 0 & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{cccccc} 2 & 2 & 6 & 1 & 0 & 0 \\ 0 & 4 & -12 & -1 & 1 & 0 \\ 0 & -12 & -20 & -2 & 0 & 1 \end{array}\right] \begin{array}{l} R_2 \to R_2 - R_1 \\ R_3 \to R_3 - 2R_1 \end{array}$$

$$\sim \left[\begin{array}{cccccc} 2 & 2 & 6 & 1 & 0 & 0 \\ 0 & 4 & -12 & -1 & 1 & 0 \\ 0 & 0 & -56 & -5 & 3 & 1 \end{array}\right] R_3 \to R_3 + 3R_2$$

$$\sim \left[\begin{array}{cccccc} 1 & 1 & 3 & \frac{1}{2} & 0 & 0 \\ 0 & 1 & -3 & \frac{-1}{4} & \frac{1}{4} & 0 \\ 0 & 0 & 1 & \frac{5}{56} & \frac{-3}{56} & \frac{-1}{56} \end{array}\right] \begin{array}{l} R_1 \to R_1 \div 2 \\ R_2 \to R_2 \div 4 \\ R_3 \to R_3 \div (-56) \end{array}$$

$$\sim \left[\begin{array}{cccccc} 1 & 1 & 0 & \frac{13}{56} & \frac{9}{56} & \frac{3}{56} \\ 0 & 1 & 0 & \frac{1}{56} & \frac{5}{56} & \frac{-3}{56} \\ 0 & 0 & 1 & \frac{5}{56} & \frac{-3}{56} & \frac{-1}{56} \end{array}\right] \begin{array}{l} R_1 \to R_1 - 3R_3 \\ R_2 \to R_2 + 3R_3 \end{array}$$

$$\sim \left[\begin{array}{cccccc} 1 & 0 & 0 & \frac{12}{56} & \frac{4}{56} & \frac{6}{56} \\ 0 & 1 & 0 & \frac{1}{56} & \frac{5}{56} & \frac{-3}{56} \\ 0 & 0 & 1 & \frac{5}{56} & \frac{-3}{56} & \frac{-1}{56} \end{array}\right]$$

$\therefore$ The inverse of the given matrix A is

$$\begin{bmatrix} \frac{12}{56} & \frac{4}{56} & \frac{6}{56} \\ \frac{1}{56} & \frac{5}{56} & \frac{-3}{56} \\ \frac{5}{56} & \frac{-3}{56} & \frac{-1}{56} \end{bmatrix} = \frac{1}{56}\begin{bmatrix} 12 & 4 & 6 \\ 1 & 5 & -3 \\ 5 & -3 & -1 \end{bmatrix}$$