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Momentum and Energy equations

The Momentum Equations:

The differential forms of the equations of motion in the velocity boundary layer are obtained by applying Newton's second law of motion to a differential control volume element in the boundary layer.

Newton's second law is an expression for momentum balance and can be stated as the net force acting on the control volume is equal to the mass times the acceleration of the fluid element within the control volume, which is also equal to the net rate of momentum outflow from the control volume.

- The forces acting on the control volume consist of body forces that act throughout the entire body of the control volume (such as gravity, electric, and magnetic forces) and are proportional to the volume of the body, and surface forces that act on the control surface (such as the pressure forces due to hydrostatic pressure and shear stresses due to viscous effects) and are proportional to the surface area.
- The surface forces appear as the control volume is isolated from its surroundings for analysis, and the effect of the detached body is replaced by a force at that location.
- pressure represents the compressive force applied on the fluid element by the surrounding fluid, and is always directed to the surface.

- Newton's second law of motion for the control volume is

$$(\text{Mass})\begin{pmatrix}\text{Acceleration} \\ \text{in a specified direction}\end{pmatrix} = \begin{pmatrix}\text{Net force (body and surface)} \\ \text{acting in that direction}\end{pmatrix} \qquad (6\text{–}22)$$

- where the mass of the fluid element within the control volume is

$$\delta m = \rho(dx \cdot dy \cdot 1)$$

- Noting that flow is steady and two-dimensional and thus u 5 u(x, y), the total differential of u is

$$du = \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy$$

- Then the acceleration of the fluid element in the x direction becomes

$$a_x = \frac{du}{dt} = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt} = u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}$$

- The forces acting on a surface are due to pressure and viscous effects.
- In two-dimensional flow, the viscous stress at any point on an imaginary surface within the fluid can be resolved into two perpendicular components:
  - One normal to the surface called Normal Stress
  - Another along the wall surface called Shear Stress.

- The normal stress is related to the velocity gradients $\partial u/\partial x$ and $\partial v/\partial y$,
- that are much smaller than $\partial u/\partial y$, to which shear stress is related.
- Then the net surface force acting in the x-direction becomes

$$F_{\text{surface},\,x} = \left(\frac{\partial \tau}{\partial y}dy\right)(dx\cdot 1) - \left(\frac{\partial P}{\partial x}dx\right)(dy\cdot 1) = \left(\frac{\partial \tau}{\partial y} - \frac{\partial P}{\partial x}\right)(dx\cdot dy\cdot 1)$$

$$= \left(\mu\frac{\partial^2 u}{\partial y^2} - \frac{\partial P}{\partial x}\right)(dx\cdot dy\cdot 1)$$

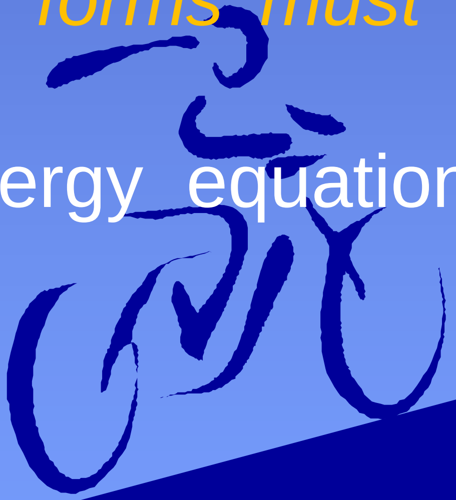
since $\tau = \mu(\partial u/\partial y)$.
dividing by $dx \cdot dy \cdot 1$

$$\rho\left(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right) = \mu\frac{\partial^2 u}{\partial y^2} - \frac{\partial P}{\partial x}$$

This is the relation for the momentum balance in the x-direction, and is known as the x-momentum equation.

If there is a body force acting in the x-direction, it can be added to the right side of the equation provided that it is expressed per unit volume of the fluid.

- Conservation of Energy Equation The energy balance for any system undergoing any process is expressed as Ein - Eout = ΔEsystem, which states that the *change in the energy content of a system during a process is equal to the difference between the energy input and the energy output*.
- During a steady-flow process, the total energy content of a control volume remains constant (and thus Δ Esystem = 0), and *the amount of energy entering a control volume in all forms must be equal to the amount of energy leaving it.*
- Then the rate form of the general energy equation reduces for a steady-flow process to Ein - Eout = 0.



- Noting that energy can be transferred by heat, work, and mass only, the energy balance for a steady-flow control volume can be written explicitly as

$$(\dot{E}_{in} - \dot{E}_{out})_{\text{by heat}} + (\dot{E}_{in} - \dot{E}_{out})_{\text{by work}} + (\dot{E}_{in} - \dot{E}_{out})_{\text{by mass}} = 0$$

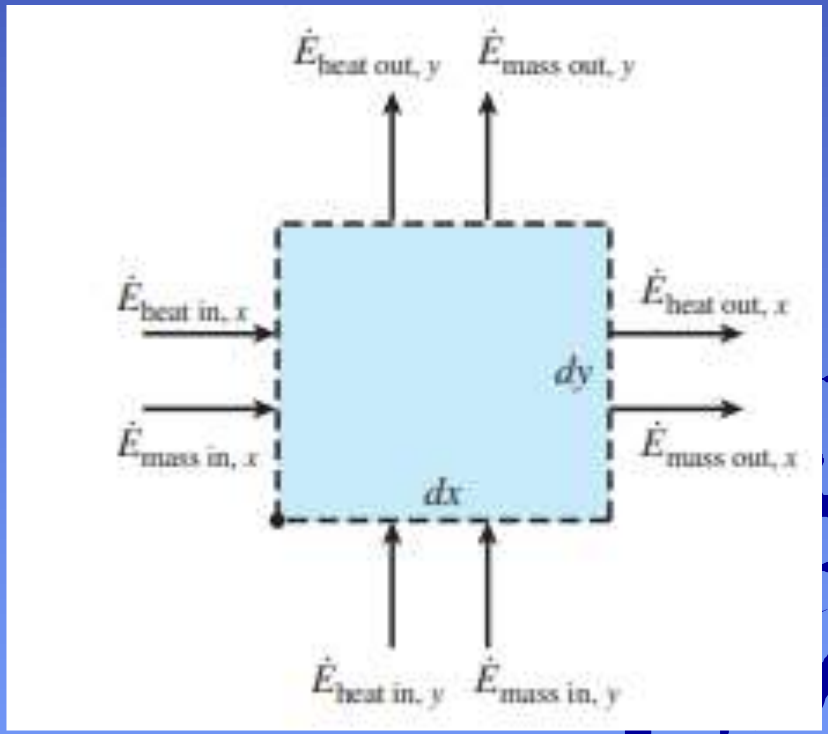
- Energy is a scalar quantity, and thus energy interactions in all directions can be combined in one equation.

- Mass flow rate of the fluid entering the control volume from the left is ρu(dy·1), the rate of energy transfer to the control volume by mass in the x-direction is, from

$$(\dot{E}_{in} - \dot{E}_{out})_{by\ mass,\ x} = (\dot{m}e_{stream})_x - \left[(\dot{m}e_{stream})_x + \frac{\partial(\dot{m}e_{stream})_x}{\partial x}dx\right]$$

$$= -\frac{\partial[\rho u(dy \cdot 1)c_p T]}{\partial x}dx = -\rho c_p\left(u\frac{\partial T}{\partial x} + T\frac{\partial u}{\partial x}\right)dxdy \quad \text{(6–31)}$$

- The energy transfers by heat and mass flow associated with a differential control volume in the thermal boundary layer in steady two- dimensional flow.



- Repeating this for the y-direction and adding the results, the net rate of energy transfer to the control volume by mass is determined to be

$$(\dot{E}_{\text{in}} - \dot{E}_{\text{out}})_{\text{by mass}} = -\rho c_p\left(u\frac{\partial T}{\partial x} + T\frac{\partial u}{\partial x}\right)dxdy - \rho c_p\left(v\frac{\partial T}{\partial y} + T\frac{\partial v}{\partial y}\right)dxdy$$

$$= -\rho c_p\left(u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y}\right)dxdy \quad \textbf{(6–32)}$$

since $\partial u/\partial x + \partial v/\partial y = 0$ from the continuity equation.

- The net rate of heat conduction to the volume element in the x-direction is

$$(\dot{E}_{in} - \dot{E}_{out})_{by\ heat,\ x} = \dot{Q}_x - \left(\dot{Q}_x + \frac{\partial Q_x}{\partial x} dx\right) = -\frac{\partial}{\partial x}\left(-k(dy \cdot 1)\frac{\partial T}{\partial x}\right)dx$$

$$= k\frac{\partial^2 T}{\partial x^2} dxdy \qquad \textbf{(6–33)}$$

- Repeating this for the y-direction and adding the results, the net rate of energy transfer to the control volume by heat conduction becomes

$$(\dot{E}_{in} - \dot{E}_{out})_{by\ heat} = k\frac{\partial^2 T}{\partial x^2}dxdy + k\frac{\partial^2 T}{\partial y^2}dxdy = k\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}\right)dxdy \qquad \textbf{(6–34)}$$

- Then the energy equation for the *steady two-dimensional flow of a fluid with constant properties and negligible shear stresses* is obtained by

$$\rho c_p\left(u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}\right)=k\left(\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}\right)$$

- which states that the net energy convected by the fluid out of the control volume is equal to the net energy transferred into the control volume by heat conduction.

- When the viscous shear stresses are not negligible, their effect is accounted for by expressing the energy equation as

$$\rho c_p\left(u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}\right)=k\left(\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}\right)+\mu\Phi$$

- where the viscous dissipation function Φ is obtained after a lengthy analysis

$$\Phi=2\left[\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial v}{\partial y}\right)^2\right]+\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)^2$$

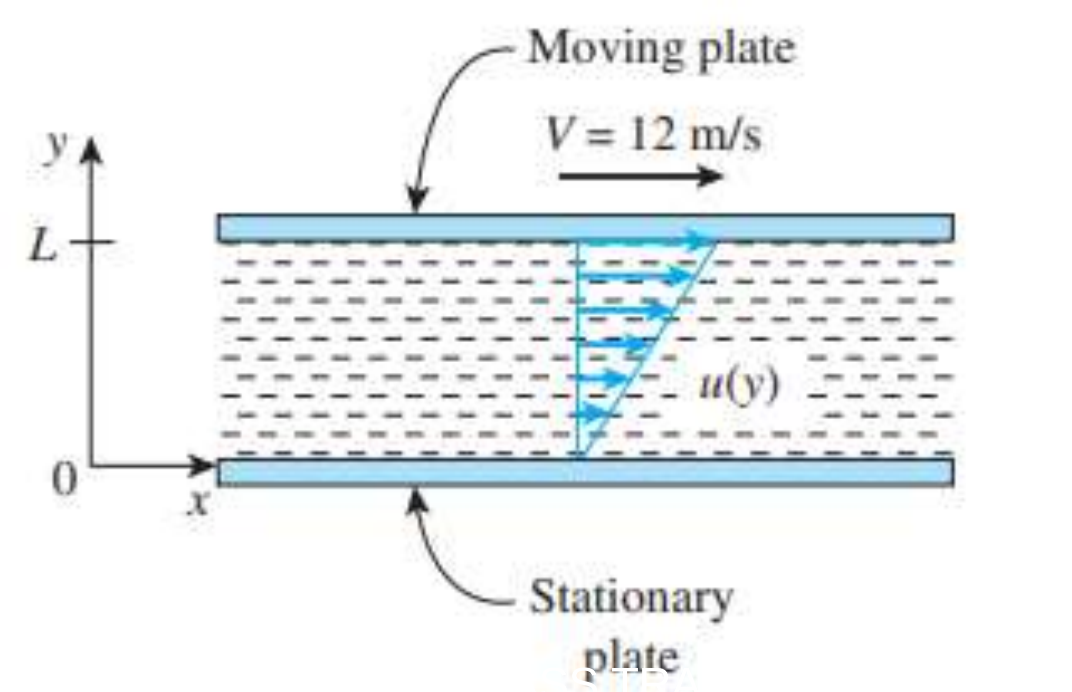
- For the special case of a stationary fluid, u = v = 0, the energy equation reduces, as expected, to the steady two-dimensional heat conduction equation,

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0$$

EXAMPLE 6–2 **Temperature Rise of Oil in a Journal Bearing**

The flow of oil in a journal bearing can be approximated as parallel flow between two large plates with one plate moving and the other stationary. Such flows are known as Couette flow.

Consider two large isothermal plates separated by 2-mm-thick oil film. The upper plates moves at a constant velocity of 12 m/s, while the lower plate is stationary. Both plates are maintained at 20°C. (*a*) Obtain relations for the velocity and temperature distributions in the oil. (*b*) Determine the maximum temperature in the oil and the heat flux from the oil to each plate (Fig. 6–32).



SOLUTION Parallel flow of oil between two plates is considered. The velocity and temperature distributions, the maximum temperature, and the total heat transfer rate are to be determined.

Assumptions **1** Steady operating conditions exist. **2** Oil is an incompressible substance with constant properties. **3** Body forces such as gravity are negligible. **4** The plates are large so that there is no variation in the z direction.

Properties The properties of oil at 20°C are (Table A–13):

$$k = 0.145 \text{ W/m·K} \quad \text{and} \quad \mu = 0.8374 \text{ kg/m·s} = 0.8374 \text{ N·s/m}^2$$

Analysis (*a*) We take the *x-axis* to be the flow direction, and *y* to be the normal direction. This is parallel flow between two plates, and thus $v = 0$. Then the continuity equation (Eq. 6–21) reduces to

Continuity: $$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \rightarrow \frac{\partial u}{\partial x} = 0 \quad \rightarrow u = u(y)$$

Therefore, the x-component of velocity does not change in the flow direction (i.e., the velocity profile remains unchanged). Noting that $u = u(y)$, $v = 0$, and $\partial P/\partial x = 0$ (flow is maintained by the motion of the upper plate rather than the pressure gradient), the x-momentum equation (Eq. 6–28) reduces to

$$\textit{x-momentum:} \qquad \rho\left(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right) = \mu\frac{\partial^2 u}{\partial y^2} - \frac{\partial P}{\partial x} \quad \rightarrow \quad \frac{d^2u}{dy^2} = 0$$

This is a second-order ordinary differential equation, and integrating it twice gives

$$u(y) = C_1 y + C_2$$

The fluid velocities at the plate surfaces must be equal to the velocities of the plates because of the no-slip condition. Therefore, the boundary conditions are $u(0) = 0$ and $u(L) = V$, and applying them gives the velocity distribution to be

$$u(y) = \frac{y}{L}V$$

Frictional heating due to viscous dissipation in this case is significant because of the high viscosity of oil and the large plate velocity. The plates are isothermal and there is no change in the flow direction, and thus the temperature depends on y only, $T = T(y)$. Also, $u = u(y)$ and $v = 0$. Then the energy equation with dissipation (Eqs. 6–36 and 6–37) reduce to

$$\textit{Energy:} \qquad 0 = k\frac{\partial^2 T}{\partial y^2} + \mu\left(\frac{\partial u}{\partial y}\right)^2 \quad \rightarrow \quad k\frac{d^2 T}{dy^2} = -\mu\left(\frac{V}{L}\right)^2$$

since $\partial u/\partial y = V/L$. Dividing both sides by k and integrating twice give

$$T(y) = -\frac{\mu}{2k}\left(\frac{y}{L}V\right)^2 + C_3 y + C_4$$

Applying the boundary conditions $T(0) = T_0$ and $T(L) = T_0$ gives the temperature distribution to be

$$T(y) = T_0 + \frac{\mu V^2}{2k}\left(\frac{y}{L} - \frac{y^2}{L^2}\right)$$

(*b*) The temperature gradient is determined by differentiating $T(y)$ with respect to y,

$$\frac{dT}{dy} = \frac{\mu V^2}{2kL}\left(1 - 2\frac{y}{L}\right)$$

The location of maximum temperature is determined by setting $dT/dy = 0$ and solving for y,

$$\frac{dT}{dy} = \frac{\mu V^2}{2kL}\left(1 - 2\frac{y}{L}\right) = 0 \quad \rightarrow \quad y = \frac{L}{2}$$

Therefore, maximum temperature occurs at mid plane, which is not surprising since both plates are maintained at the same temperature. The maximum temperature is the value of temperature at $y = L/2$,

$$T_{\max} = T\left(\frac{L}{2}\right) = T_0 + \frac{\mu V^2}{2k}\left(\frac{L/2}{L} - \frac{(L/2)^2}{L^2}\right) = T_0 + \frac{\mu V^2}{8k}$$

$$= 20 + \frac{(0.8374\ \text{N·s/m}^2)(12\ \text{m/s})_2}{8(0.145\ \text{W/m·K})}\left(\frac{1\ \text{W}}{1\ \text{N·m/s}}\right) = 124°\text{C}$$

Heat flux at the plates is determined from the definition of heat flux,

$$\dot{q}_0 = -k\frac{dT}{dy}\bigg|_{y=0} = -k\frac{\mu V^2}{2kL}\bigg|(1-0) = -\frac{\mu V^2}{2L}$$

$$= -\frac{(0.8374\ \text{N·s/m}^2)(12\ \text{m/s})^2}{2(0.002\ \text{m})}\left(\frac{1\ \text{kW}}{1000\ \text{N·m/s}}\right) = \mathbf{-30.1\ kW/m^2}$$

$$\dot{q}_L = -k\frac{dT}{dy}\bigg|_{y=L} = -k\frac{\mu V^2}{2kL}(1-2) = \frac{\mu V^2}{2L} = -\dot{q}_0 = \mathbf{30.1\ kW/m^2}$$

Therefore, heat fluxes at the two plates are equal in magnitude but opposite in sign.

ANALOGIES BETWEEN MOMENTUM AND HEAT TRANSFER

- In forced convection analysis, we are primarily interested in the determination of the quantities Cf (to calculate shear stress at the wall) and Nu (to calculate heat transfer rates).
- Therefore, it is very desirable to have a relation between Cf and Nu so that we can calculate one when the other is available. Such relations are developed on the basis of the similarity between momentum and heat transfers in boundary layers, and are known as Reynolds analogy and Colburn analogy.

Reynolds – Colburn Analogy

- Reconsider the non-dimensionalized momentum and energy equations for steady, incompressible, laminar flow of a fluid with constant properties and negligible viscous dissipation

Momentum:
$$u^*\frac{\partial u^*}{\partial x^*} + v^*\frac{\partial u^*}{\partial y^*} = \frac{1}{Re_L}\frac{\partial^2 u^*}{\partial y^{*2}}$$

Energy:
$$u^*\frac{\partial T^*}{\partial x^*} + v^*\frac{\partial T^*}{\partial y^*} = \frac{1}{Re_L}\frac{\partial^2 T^*}{\partial y^{*2}}$$

- which are exactly of the same form for the dimensionless velocity u* and temperature T*. The boundary conditions for u* and T* are also identical.
- Therefore, the functions u* and T* must be identical, and thus the first derivatives of u* and T* at the surface must be equal to each other,

$$\left.\frac{\partial u^*}{\partial y^*}\right|_{y^*=0} = \left.\frac{\partial T^*}{\partial y^*}\right|_{y^*=0}$$

- Then we have

$$C_{f,x}\frac{\mathrm{Re}_L}{2} = \mathrm{Nu}_x \quad (\mathrm{Pr} = 1)$$

Where,

$$\mathrm{St} = \frac{h}{\rho c_p V} = \frac{\mathrm{Nu}}{\mathrm{Re}_L \mathrm{Pr}}$$

- Reynolds analogy is of limited use because of the restrictions Pr = 1 and δP*/ δ x* = 0 on it, and it is desirable to have an analogy that is applicable over a wide range of Pr. This is done by adding a Prandtl number correction.
- The friction coefficient and Nusselt number for a flat plate were determined in

$$C_{f,x} = 0.664\, Re_x^{-1/2} \quad \text{and} \quad Nu_x = 0.332\, Pr^{1/3}\, Re_x^{1/2}$$

- Taking their ratio and rearranging give the desired relation, known as the ***Modified Reynolds Analogy Or Colburn Analogy***

EXAMPLE 6–3 **Finding Convection Coefficient from Drag Measurement**

A 2-m × 3-m flat plate is suspended in a room, and is subjected to air flow parallel to its surfaces along its 3-m-long side. The free stream temperature and velocity of air are 20°C and 7 m/s. The total drag force acting on the plate is measured to be 0.86 N. Determine the average convection heat transfer coefficient for the plate (Fig. 6–42).

SOLUTION A flat plate is subjected to air flow, and the drag force acting on it is measured. The average convection coefficient is to be determined.

Assumptions **1** Steady operating conditions exist. **2** The edge effects are negligible. **3** The local atmospheric pressure is 1 atm.

Properties The properties of air at 20°C and 1 atm are (Table A–15):

$$\rho = 1.204 \text{ kg/m}^3, \qquad c_p = 1.007 \text{ kJ/kg·K}, \qquad \text{Pr} = 0.7309$$

Analysis The flow is along the 3-m side of the plate, and thus the characteristic length is $L = 3$ m. Both sides of the plate are exposed to air flow, and thus the total surface area is

$$A_s = 2WL = 2(2\text{ m})(3\text{ m}) = 12\text{ m}^2$$

For flat plates, the drag force is equivalent to friction force. The average friction coefficient C_f can be determined from Eq. 6–11,

$$F_f = C_f A_s \frac{\rho V^2}{2}$$

Solving for C_f and substituting,

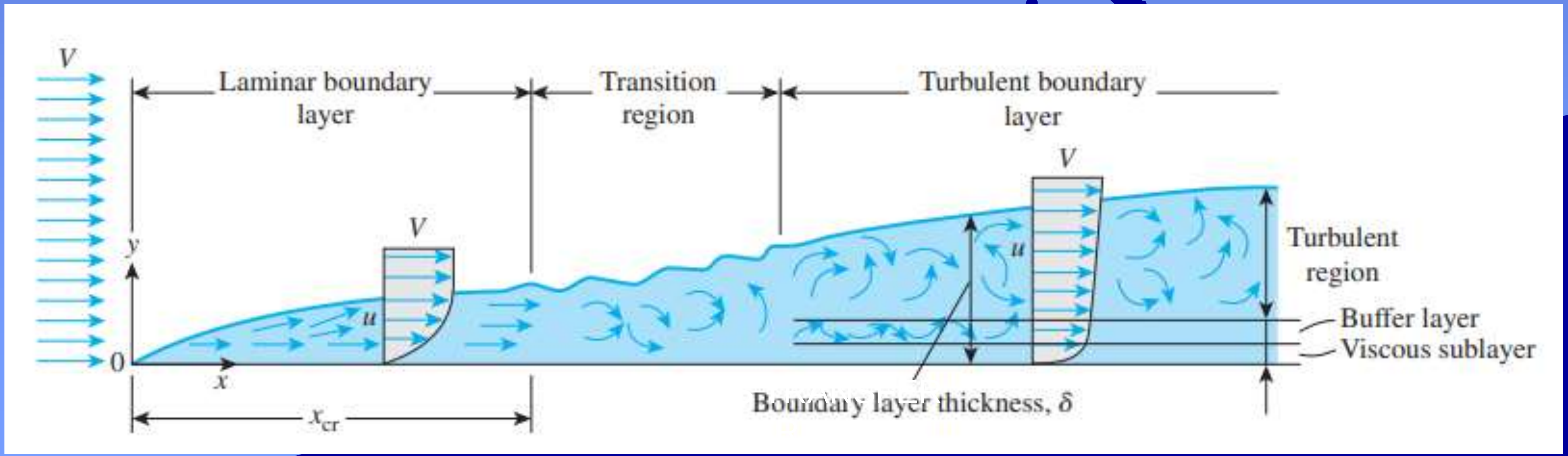
$$C_f = \frac{F_f}{\rho A_s V^2/2} = \frac{0.86\text{ N}}{(1.204\text{ kg/m}^3)(12\text{ m}^2)(7\text{ m/s})^2/2}\left(\frac{1\text{ kg}\cdot\text{m/s}^2}{1\text{ N}}\right) = 0.00243$$

Then the average heat transfer coefficient can be determined from the modified Reynolds analogy (Eq. 6–83) to be

$$h = \frac{C_f \rho V c_p}{2\text{ Pr}^{2/3}} = \frac{0.00243}{2}\frac{(1.204\text{ kg/m}^3)(7\text{ m/s})(1007\text{ J/kg}\cdot\text{K})}{0.7309^{2/3}} = \mathbf{12.7\text{ W/m}^2\cdot\text{K}}$$

TURBULENT BOUNDARY LAYER

- Consider the parallel flow of a fluid over a flat plate,. Surfaces that are slightly contoured such as turbine blades can also be approximated as flat plates with reasonable accuracy.
- The x-coordinate is measured along the plate surface from the leading edge of the plate in the direction of the flow, and y is measured from the surface in the normal direction.
- The fluid approaches the plate in the x-direction with a uniform velocity V, which is practically identical to the free-stream velocity over the plate away from the surface



- The velocity of the particles in the first fluid layer adjacent to the plate becomes zero because of the no-slip condition.
- This motionless layer slows down the particles of the neighboring fluid layer as a result of friction between the particles of these two adjoining fluid layers at different velocities.
- As a result, the x-component of the fluid velocity, u, varies from 0 at y = 0 to nearly V at y = δ.

- The region of the flow above the plate bounded by δ in which the effects of the viscous shearing forces caused by fluid viscosity are felt is called the velocity boundary layer.
- The boundary layer thickness, δ, is typically defined as the distance y from the surface at which u 5 0.99V.
- The hypothetical line of u = 0.99V divides the flow over a plate into two regions: the boundary layer region, in which the viscous effects and the velocity changes are significant, and the Irrotational flow region, in which the frictional effects are negligible and the velocity remains essentially constant.

MIXING LENGTH CONCEPT

- Mixing length is defined as that distance in the transverse direction which must be covered by a lump of fluid particle travelling with its original mean velocity in order to make the difference between its velocity and the velocity of the new layer equal to the mean transverse fluctuation in the turbulent flow.
- The concept of mixing length is similar to the mean free path used in kinetic theory of gases. Prandtl proposed that when there is mixing between two fluid elements; there is complete exchange of momentum.

- The turbulent shear stress can only be calculated when the fluctuating components are known. But since they are very difficult to measure Prandtl presented the mixing length theory which can measure turbulent shear stress in terms of measurable quantity.

- Consider two fluid masses separated by mixing length (l).
- The lower mass has an instantaneous velocity $\bar{u}$ and a velocity fluctuation.
- The upper mass has an instantaneous Velocity $\bar{u}+u'\bar{u}+u'$ but it does not have any velocity fluctuation in the Y direction.
- By virtue of velocity fluctuation v' the lower mass moves towards the upper mass and exchanges its momentum completely.

For the lower mass:

The initial velocity= $\bar{u}$

The initial momentum = $(\rho A v')\bar{u}$

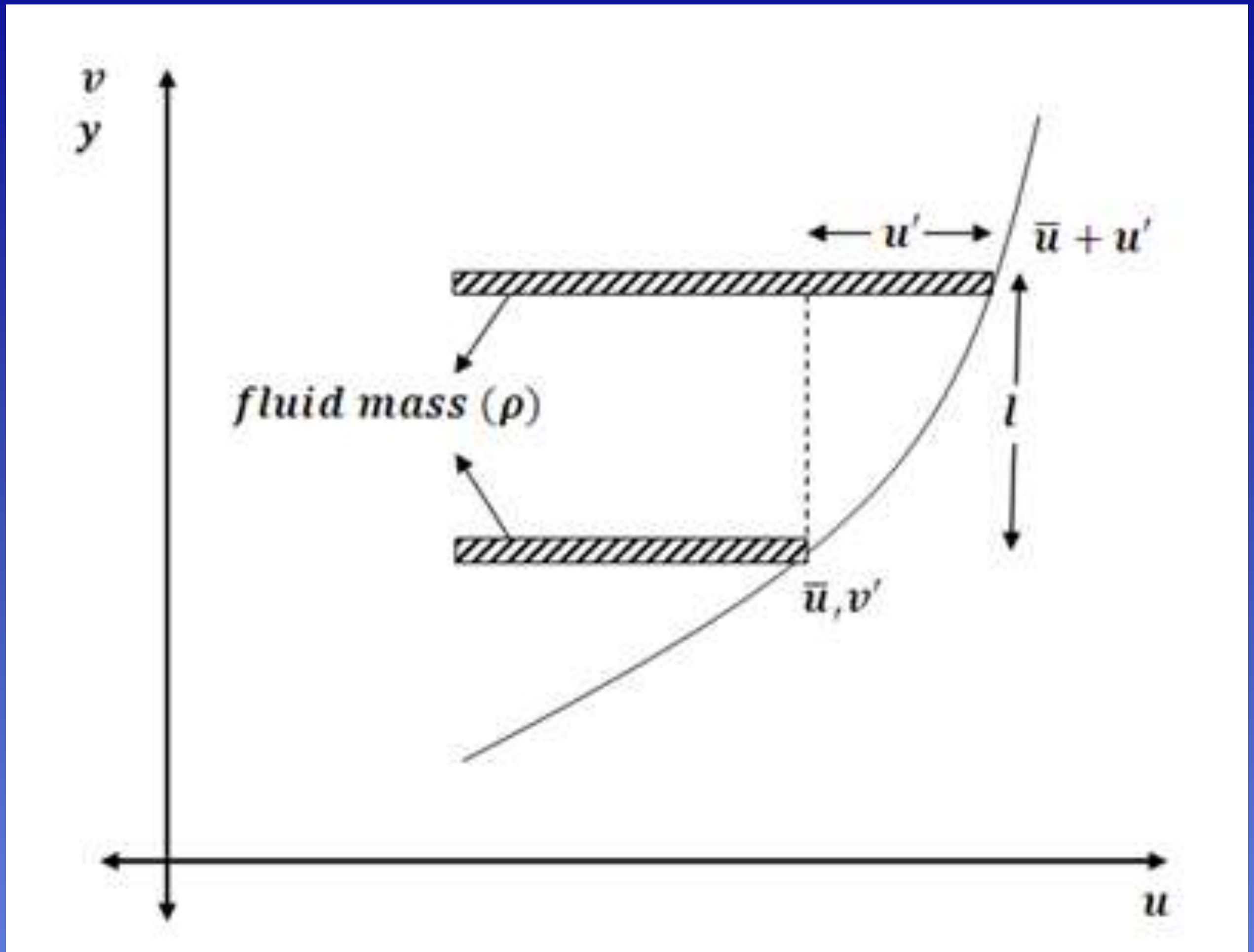
The final momentum of the same mass =$(\rho A v')(\bar{u} + u')$

The change in momentum = $(\rho A v')(\bar{u} + u') - (\rho A v')\bar{u}$

∴Shear force =$\rho A v' v'$

∴Shearstress= $\frac{shearingforce}{shearingarea} = \frac{\rho A u' v'}{A} = \rho u' v'$

This is called Turbulent Reynolds stress.



As can be seen in the figure,

$$u' = l\frac{du}{dy}$$

$$v' = l\frac{du}{dy}$$

$$\therefore u'v' = l^2\left(\frac{du}{dy}\right)^2$$

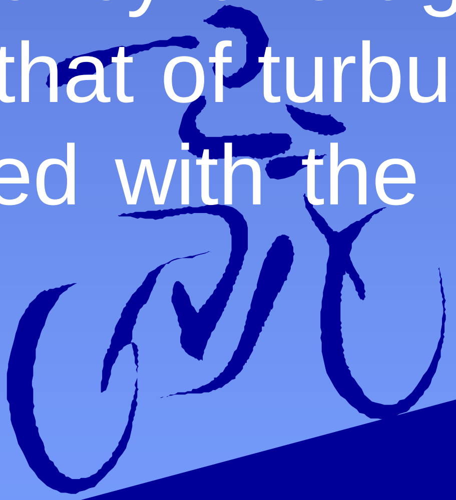
$$\therefore \text{Shearstress} = \rho u'v' = \rho l^2\left(\frac{du}{dy}\right)^2$$

Turbulence Modelling

- The instantaneous variables are decomposed into MEAN and FLUCTUATING quantities:

$$U_i = \overline{U}_i + u_i \; ; \quad P = \overline{P} + p \; ; \quad H = \overline{H} + h \; ; \quad C = \overline{C} + c$$

- where the mean values are obtained by averaging over a time scale, dt, which is long compared to that of turbulent motion, and in unsteady problems small compared with the unsteadiness of the mean motion.



- The definitions of instantaneous quantities are substituted into the equations of the INSTANTANEOUS MOTION, which are then averaged to produce the equations of the MEAN MOTION.
- For an incompressible flow, the AVERAGED equations are:

$$\sum \frac{\partial U_i}{\partial x_i} = 0$$

$$\rho \frac{\partial U_i}{\partial t} + \rho \sum_j \frac{\partial \left(U_i U_j\right)}{\partial x_j} = -\frac{\partial P}{\partial x_i} + B_i + \rho \sum_j \frac{\partial}{\partial x_j}\left[\nu_L\left(\frac{\partial U_i}{\partial x_j}+\frac{\partial U_j}{\partial x_i}\right) - \overline{u_i u_j}\right]$$

$$\rho \frac{\partial C}{\partial t} + \rho\sum_j \frac{\partial\left(U_j C\right)}{\partial x_j} = \rho \sum_j \frac{\partial}{\partial x_j}\left(\frac{\nu_L}{Pr_L(C)}\frac{\partial C}{\partial x_j} - \overline{u_i c}\right)$$

$$\rho \frac{\partial H}{\partial t} + \rho\sum_j \frac{\partial\left(U_j H\right)}{\partial x_j} = \rho\left[\frac{\nu_L}{Pr_L(H)}\frac{\partial H}{\partial x_j} - \overline{u_i h}\right] + S$$

- The statistical-averaging process has introduced unknown turbulent correlations into the mean-flow equations which represent the turbulent transport of momentum, heat and mass - the REYNOLDS STRESSES and FLUXES.

In general, there are:

- 6 Reynolds stress components: $-u_i u_j$
- 3 Reynolds enthalpy flux components: $-u_i h$
- 3 Reynolds mass flux components: $-u_i c$

- A TURBULENCE MODEL can be described as a set of relations and equations needed to determine the unknown turbulent correlations that have arisen from the averaging process.
- Turbulence models of various complexity have been developed, and with very few exceptions, they can be classified as EDDY-VISCOSITY MODELS or REYNOLDS-STRESS MODELS.
- In EDDY-VISCOSITY MODELS, the unknown correlations are assumed to be proportional to the spatial gradients of the quantity they are meant to transport.
- In REYNOLDS-STRESS MODELS, the unknown correlations are determined directly from the solution of differential transport equations in which they are the dependent variables.

Classification Of Turbulence Models

- ZERO-EQUATION MODELS- V_s and L_s are calculated directly from the local mean flow quantities (e.g. Prandtl's mixing-length model).

- ONE-EQUATION MODELS- V_s is calculated from a suitable transport equation, usually the turbulent kinetic energy, KE, and the length scale, L_s, is prescribed empirically (e.g. Prandtl's k-L model).

- TWO-EQUATION MODELS- V_s and L_s are both calculated from transport equations, usually KE and its dissipation rate EP.

- REYNOLDS-STRESS/FLUX TRANSPORT MODELS- These are models involving the solution of transport equations for the Reynolds stresses and fluxes, together with a transport equation for the length scale, usually EP.

- ALGEBRAIC STRESS/FLUX MODELS- These models simplify the stress/flux transport equations to provide algebraic expressions for the turbulent correlations, which are then solved together with a 2-equation model.

k-ε model

- The k equation is:

$$\frac{\partial k}{\partial t} + U_j \frac{\partial k}{\partial x_j} = \frac{\mu_t}{\rho} S^2 - \epsilon + \frac{\partial}{\partial x_j}\left[\frac{1}{\rho}\left(\mu + \frac{\mu_t}{\sigma_k}\right)\frac{\partial k}{\partial x_j}\right]$$

- The ε equation can be obtained from the NS equations but it contains several undetermined quantities; it is therefore derived “mimicking” the k equation

$$\frac{\partial \epsilon}{\partial t} + U_j \frac{\partial \epsilon}{\partial x_j} = \frac{\epsilon}{k}\left(C_{1\epsilon}\frac{\mu_t}{\rho} S^2 - C_{2\epsilon}\epsilon\right) + \frac{\partial}{\partial x_j}\left[\frac{1}{\rho}\left(\mu + \frac{\mu_t}{\sigma_\epsilon}\right)\frac{\partial \epsilon}{\partial x_j}\right]$$

- The eddy viscosity is obtained as:

$$\mu_t = \rho C_\mu \frac{k^2}{\epsilon}$$

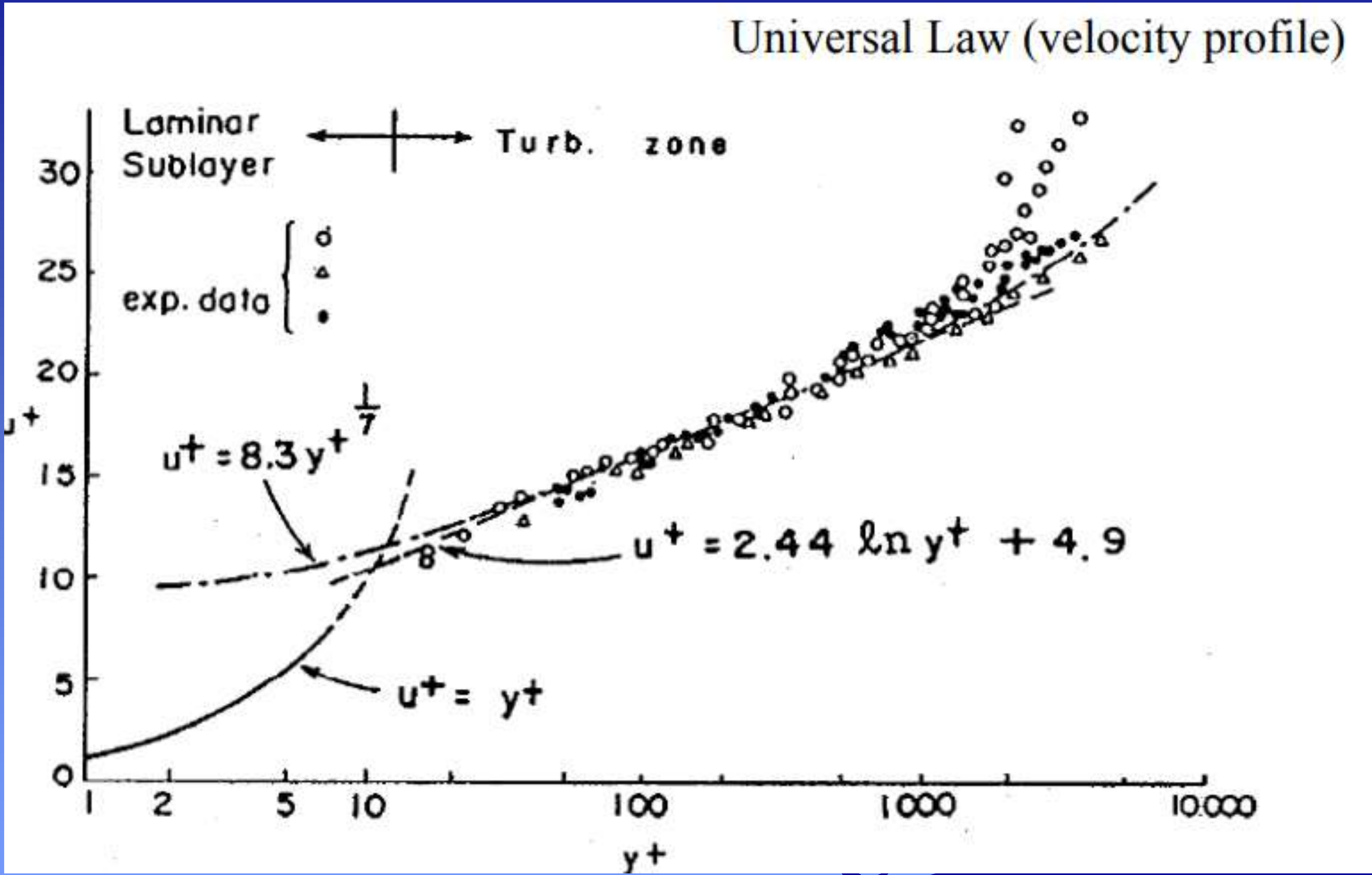
- There are 5 free constants

$$\sigma_k, \sigma_\epsilon, C_{1\epsilon}, C_{2\epsilon}, C_\mu$$

- The constants can be determined studying simple flows:

1.	Decaying homogeneous isotropic turbulence	$C_{2\epsilon};$
2.	Homogeneous shear flow	$C_{1\epsilon}$ $C_{2\epsilon};$
3.	The Logarithmic Layer	$C_{1\epsilon}$ $C_{2\epsilon}; C_\mu, \sigma_\epsilon$

Structure of the Turbulent Boundary Layer



- At High Reynolds number the viscous dominated layer is so thin that it is very difficult to resolve it

Von Karman

- the exact mathematical solution of the differential equations describing the laminar flow of a fluid over a flat surface in deriving the boundary layer thickness and the heat transfer coefficient.
- To circumvent the problems involved in solving the partial differential equations of the boundary layer, Theodore von Karman suggested the approximate integral method in which he considered a control volume that extends from the wall to beyond the boundary layer.
-

- Let us consider a control volume (*CV*) bounded by the two planes *AB* and *CD* normal to the surface, a distance *dx* apart, and a parallel plane in the free stream at a distance l.
- Let us consider unit width of plate in *z*-direction

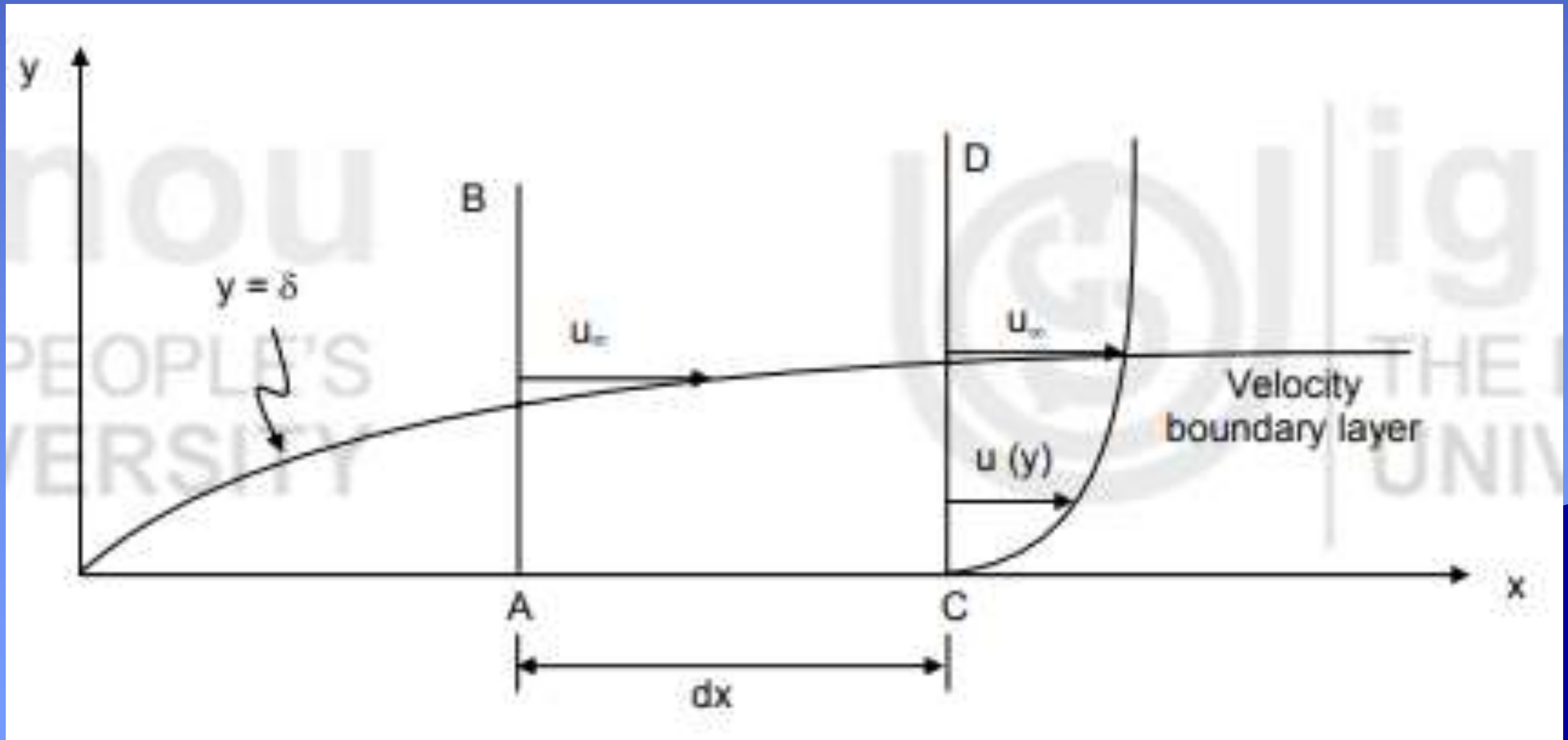


Figure 8.5 : Control Volume for Integral Momentum Conservation Analysis

- Momentum flow across face *AB* into the *CV* in is

$$= \int_0^1 \rho u^2 dy$$

- Similarly, momentum flow across face *CD* is

$$= \int_0^1 \rho u^2 dy + \frac{d}{dx}\left(\int_0^1 \rho u^2 dy\right) dx$$

- Fluid entering across *BD* at the rate

$$= \frac{d}{dx}\left(\int_0^1 \rho u dy\right) dx$$

- This quantity is the difference between the rate of flow leaving across *CD* and that entering across *AB*. Since the fluid entering across *BD* has a velocity component in the *x*-direction equal to *u*∞ , the flow of *x*-momentum across the upper face into the *CV* is

$$u_\infty \frac{d}{dx}\left(\int_0^1 \rho u dy\right) dx$$

- Net *x*-momentum transfer

I = outflow – inflow

$$I = \int_0^1 \rho u^2 dy + \frac{d}{dx}\left(\int_0^1 \rho u^2 dy\right)dx - \int_0^1 \rho u^2 dy$$

- For $y \geq \delta, u = u_\infty$ and the integrand *I* will be zero.
- We have to consider the integrand only within the limits from *y* = 0 to *y* = ∞ There will be no shear across face *BD* outside the boundary layer

where $\frac{du}{dy} = 0$

- A shear force t $_w$ acts at the fluid-solid interface, and there will be pressure forces acting on faces *AB* and *CD*. Net forces acting on the *CV* are

$$p\delta - \left(p + \frac{dp}{dx}dx\right)\delta - \tau_w dx = -\delta\frac{dp}{dx}dx - \tau_w dx$$

- By Newton's second law of motion

$$-\delta \frac{dp}{dx} dx - \tau_w dx = -\frac{d}{dx}\left[\int_0^\delta \rho u \left(u_\infty - u\right) dy\right] dx$$

- For flow over a flat plate the pressure gradient in the x-direction, $\frac{dp}{dx}$ can be neglected.

Therefore, $$\frac{d}{dx}\int_0^\delta \rho u \left(u_\infty - u\right) dy = \tau_w$$

- The above equation is often called von Karman's momentum integral equation. Assuming a four-term polynomial for the velocity distribution [3]

$$u(y) = a + by + cy^2 + dy^3$$

- Where the constants are evaluated from the boundary conditions : at

$$y = 0,\ u = 0 \text{ and so } a = 0$$

$$u = v = 0 \text{ and } \frac{\partial^2 u}{\partial y^2} = 0$$

$$y = \delta,\ u = u_\infty \text{ and } \frac{\partial u}{\partial y} = 0$$

From these conditions we find

$$a = 0, b = \frac{3}{2}\frac{u_\infty}{\delta}, c = 0, d = -\frac{u_\infty}{2\delta^3}$$

Substituting in Eq. (8.42)

$$u = \frac{3}{2}\frac{u_\infty}{\delta}y - \frac{u_\infty}{2}\frac{y^3}{\delta^3}$$

$$\frac{u}{u_\infty} = \frac{3}{2}\frac{y}{\delta} - \frac{1}{2}\left(\frac{y}{\delta}\right)^3$$

Substituting Eq. (8.44) for the velocity distribution in the integral momentum

$$\frac{d}{dx}\int_0^\delta \left(\rho u u_\infty - \rho u^2\right) dy = \tau_w$$

$$\text{L.H.S.} = \frac{d}{dx}\int_0^{\delta}\left[\rho u_\infty^2\left(\frac{3}{2}\frac{y}{\delta}-\frac{1}{2}\frac{y^3}{\delta^3}\right)-u_\infty^2\rho\left(\frac{3}{2}\frac{y}{\delta}-\frac{1}{2}\frac{y^3}{\delta^3}\right)^2\right]dy$$

$$=\frac{d}{dx}\int_0^{\delta}\left[\rho u_\infty^2\left(\frac{3}{2}\frac{y}{\delta}-\frac{1}{2}\frac{y^3}{\delta^3}\right)-\rho u_\infty^2\left(\frac{9}{4}\frac{y^2}{\delta^2}-\frac{3}{2}\frac{y^4}{\delta^4}+\frac{1}{4}\frac{y^6}{\delta^6}\right)\right]dy$$

$$=\frac{d}{dx}\left[\rho u_\infty^2\left\{\left(\frac{3}{2\delta}\frac{\delta^2}{2}-\frac{1}{2\delta^3}\frac{\delta^4}{4}\right)-\frac{9}{4\delta^2}\frac{\delta^3}{3}+\frac{3}{2\delta^4}\frac{\delta^5}{5}-\frac{1}{4\delta^6}\frac{\delta^7}{7}\right\}\right]dy$$

$$=\frac{d}{dx}\left(\rho u_\infty^2\frac{39}{280}\delta\right)$$

Again, $\tau_w = \mu\left(\frac{du}{dy}\right)_{y=0} = \mu\frac{3}{2}\frac{u_\infty}{\delta}$

Therefore, $\frac{d}{dx}\left(\rho u_\infty^2 \frac{39}{280}\delta\right) = \mu\frac{3}{2}\frac{u_\infty}{\delta}$

$$\frac{39}{280}\rho u_\infty^2 \frac{d\delta}{dx} = \frac{3}{2}\mu\frac{u_\infty}{\delta}$$

$$\delta = \left(\frac{280}{13}\frac{vx}{u_\infty}\right)^{1/2} = 4.64\left(\frac{vx}{u_\infty}\right)^{1/2}$$

When $x = 0, \delta = 0$ and $\delta \ \infty (x)^{1/2}$

Also, $\frac{\delta}{x} = \frac{4.64}{(\mathrm{Re}_x)^{1/2}}$

where local Reynolds number $\mathrm{Re}_x = \frac{u_\infty x}{v}$.

Substituting for δ from Eq. (8.50) in Eq. (8.46),

$$\tau_w = \mu \frac{3}{2} \frac{u_\infty (\mathrm{Re}_x)^{1/2}}{4.64x}$$

Dividing both sides by $\frac{1}{2}\rho u_\infty^2$

$$\frac{\tau_w}{1/2\rho u_\infty^2} = C_{f_x} = \frac{3u_\infty}{9.28} \frac{\mu(\mathrm{Re}_x)^{1/2} \times 2}{x \rho u_\infty^2}$$

$$C_{f_x} = 0.647 \frac{\nu}{u_\infty x} (\mathrm{Re}_x)^{1/2} = \frac{0.647}{(\mathrm{Re}_x)^{1/2}}$$

The exact analysis gave us (Eq. (4.91))

$$C_{f_x} = \frac{0.664}{(\mathrm{Re}_x)^{1/2}}$$

Turbulent flow in a tube

- Flow in a tube can be laminar or turbulent, depending on the flow conditions. Fluid flow is streamlined and thus laminar at low velocities, but turns turbulent as the velocity is increased beyond a critical value.
- Transition from laminar to turbulent flow does not occur suddenly; rather, it occurs over some range of velocity where the flow fluctuates between laminar and turbulent flows before it becomes fully turbulent. Most pipe flows encountered in practice are turbulent.
- Laminar flow is encountered when highly viscous fluids such as oils flow in small diameter tubes or narrow passages.

- For flow in a circular tube, the Reynolds number is defined as

$$\mathrm{Re} = \frac{V_{\text{avg}} D}{\nu} = \frac{\rho V_{\text{avg}} D}{\mu} = \frac{\rho D}{\mu}\left(\frac{\dot{m}}{\rho \pi D^2/4}\right) = \frac{4\dot{m}}{\mu \pi D}$$

- where V_{avg} is the average flow velocity, D is the diameter of the tube, and n 5 m/r is the kinematic viscosity of the fluid.
- For flow through noncircular tubes, the Reynolds number as well as the Nusselt number, and the friction factor are based on the hydraulic diameter D_h defined as

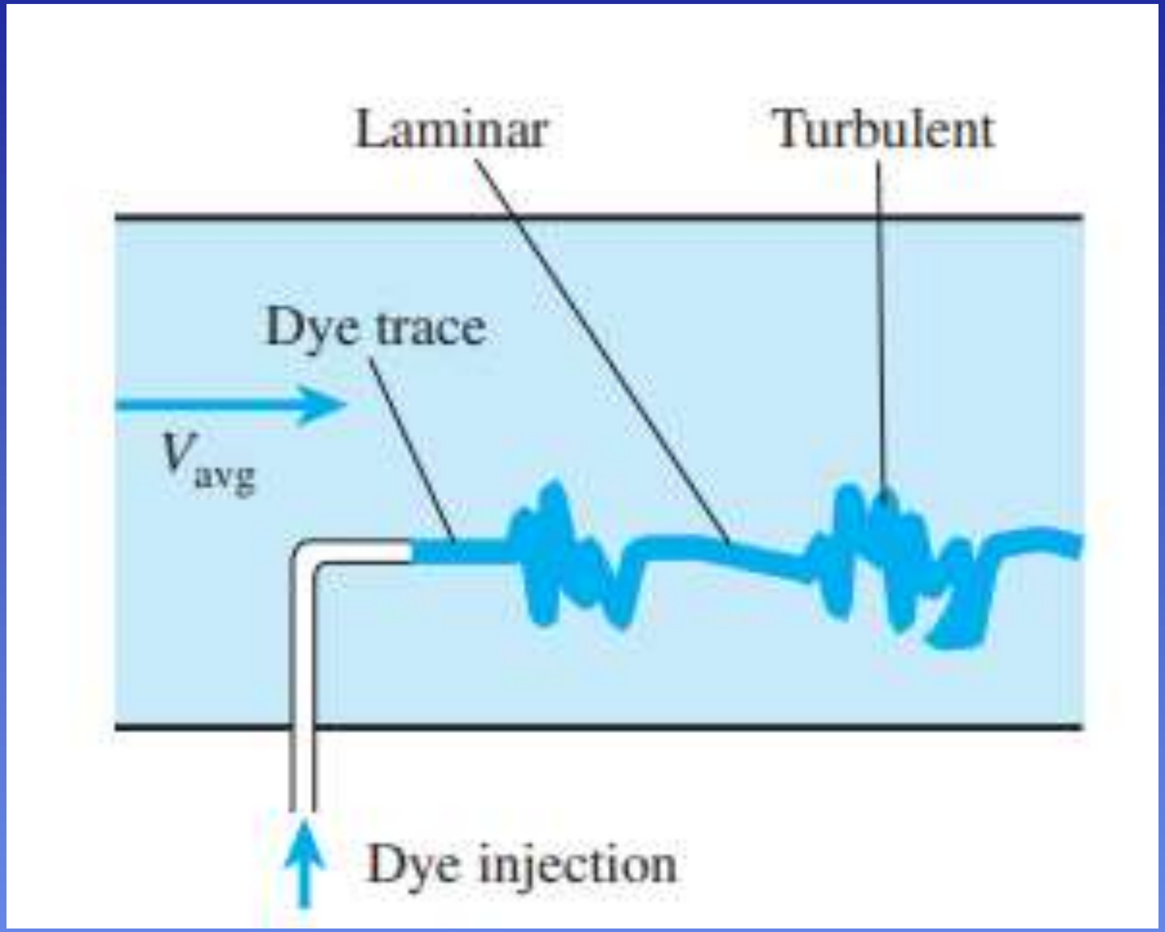
$$D_h = \frac{4A_c}{p}$$

- The hydraulic diameter is defined such that it reduces to ordinary diameter D for circular tubes since

Circular tubes: $$D_h = \frac{4A_c}{p} = \frac{4\pi D^2/4}{\pi D} = D$$

- This is because the transition from laminar to turbulent flow also depends on the degree of disturbance of the flow by surface roughness, pipe vibrations, and the fluctuations in the flow

- In transitional flow, the flow switches between laminar and turbulent in a disorderly fashion.



- It should be kept in mind that laminar flow can be maintained at much higher Reynolds numbers in very smooth pipes by avoiding flow disturbances and tube vibrations. In such carefully controlled experiments, laminar flow has been maintained at Reynolds numbers of up to 100,000.

THANK YOU