



UNIT-1 VECTOR CALCULUS

STOKE'S THEOREM

*Stokes' Theorem:*

The line integral of the tangential component of a vector function  $\vec{F}$  around a simple closed curve  $C$  is equal to the surface integral of the normal component of curl  $\vec{F}$  over an open surface  $S$ .

$$\text{ie., } \int_C \vec{F} \cdot d\vec{r} = \iint_S (\nabla \times \vec{F}) \cdot \hat{n} \, ds$$

To verify Stokes' Theorem for  $\vec{F} = (x^2 + y^2) \vec{i} - 2xy \vec{j}$  taken around the rectangle bounded by the lines  $x = \pm a$ ,  $y = 0$ ,  $y = b$ .

Soln.

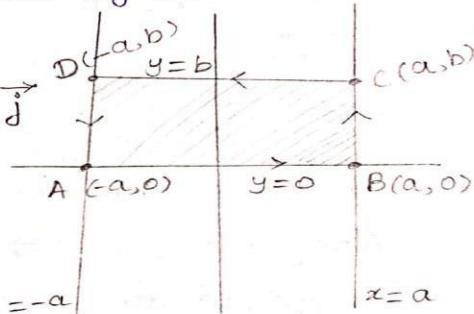
$$\text{Given } \vec{F} = (x^2 + y^2) \vec{i} - 2xy \vec{j}$$

ST

$$\int_C \vec{F} \cdot d\vec{r} = \iint_S \nabla \times \vec{F} \cdot \hat{n} \, ds$$

Now,

$$\begin{aligned} \nabla \times \vec{F} &= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 + y^2 & -2xy & 0 \end{vmatrix} \\ &= \vec{i}[0-0] - \vec{j}[0-0] + \vec{k}[-ay - 2y] \\ &= -4y \vec{k} \end{aligned}$$



RHS

$$\begin{aligned} \iint_S \nabla \times \vec{F} \cdot \hat{n} \, ds &= \iint_S (-4y \vec{k}) \cdot \vec{k} \, dx \, dy \\ &= \iint_S (-4y) \, dx \, dy \\ &= -4 \int_0^b \int_{-a}^a y \, dx \, dy \\ &= -4 \int_0^b y \left[ x \right]_{-a}^a \, dy \end{aligned}$$





$$\begin{aligned}
 &= -4 \int_0^b y [a+a] dy \\
 &= -8a \int_0^b y dy \\
 &= -8a \left[ \frac{y^2}{2} \right]_0^b
 \end{aligned}$$

$$\iint_S (\nabla \times \vec{F}) \cdot \hat{n} ds = -4ab^2 \rightarrow (1)$$

Given  $\vec{F} = (x^2 + y^2) \vec{i} - 2xy \vec{j}$

$$d\vec{r} = dx \vec{i} + dy \vec{j} + dz \vec{k}$$

$$\vec{F} \cdot d\vec{r} = (x^2 + y^2) dx - 2xy dy$$

LHS:

$$\int_C \vec{F} \cdot d\vec{r} = \int_{AB} + \int_{BC} + \int_{CD} + \int_{DA}$$

Along AB [ $y=0 \Rightarrow dy=0$ ]

$$\begin{aligned}
 \int_{AB} (x^2 + y^2) dx - 2xy dy &= \int_{-a}^a x^2 dx \\
 &= \left( \frac{x^3}{3} \right)_a^{-a} \\
 &= \frac{a^3}{3} - \frac{(-a)^3}{3} \\
 &= \frac{2a^3}{3}
 \end{aligned}$$

$$\int_{AB} (x^2 + y^2) dx - 2xy dy = \frac{2a^3}{3}$$

Along BC [ $x=a \Rightarrow dx=0$ ]

$$\begin{aligned}
 \int_{BC} (x^2 + y^2) dx - 2xy dy &= \int_0^b [0 - 2ay dy] \\
 &= -2a \int_0^b y dy
 \end{aligned}$$





UNIT-1 VECTOR CALCULUS

STOKE'S THEOREM

$$= -2a \left[ \frac{y^2}{2} \right]_0^b$$

$$\int_{BC} (x^2 + y^2) dx - 2xy dy = -ab^2$$

Along CD [ $-y = b \Rightarrow dy = 0$ ]

$$\int_{CD} (x^2 + y^2) dx - 2xy dy = \int_a^{-a} (x^2 + b^2) dx$$

$$= \left[ \frac{x^3}{3} + b^2 x \right]_a^{-a}$$

$$= \left( \frac{a^3}{3} - ab^2 \right) - \left( \frac{a^3}{3} + ab^2 \right)$$

$$= -2ab^2 - \frac{2a^3}{3}$$

Along DA ( $x = -a \Rightarrow dx = 0$ )

$$\int_{DA} (x^2 + y^2) dx - 2xy dy = \int_b^0 0 - 2(-a)y dy$$

$$= \int_b^0 2ay dy$$

$$= 2a \left[ \frac{y^2}{2} \right]_b^0$$

$$= 0 - ab^2$$

$$= -ab^2$$

$$\begin{aligned} \therefore \int_C \vec{F} \cdot d\vec{r} &= \int_{AB} + \int_{BC} + \int_{CD} + \int_{DA} \\ &= \frac{2a^3}{3} - ab^2 - 2ab^2 - \frac{2a^3}{3} - ab^2 \\ &= -4ab^2 \end{aligned}$$

$$\text{LHS} = \text{RHS}$$

From (1) & (2), hence Stoke's theorem is verified.

