



2]. verify Stoke's theorem $\vec{F} = (x^2 - y^2)\vec{i} + 2xy\vec{j}$
in the rectangular region bounded by $x=0, x=a, y=0, y=b$.

Soln.

ST

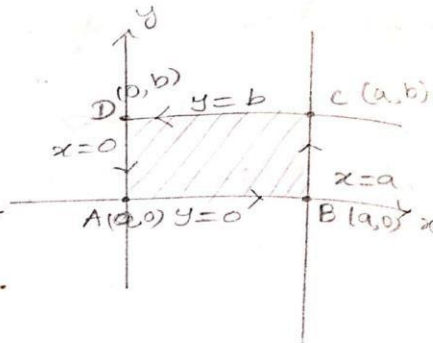
$$\int_C \vec{F} \cdot d\vec{r} = \iint_S (\nabla \times \vec{F}) \cdot \hat{n} \, ds$$

Given $\vec{F} = (x^2 - y^2)\vec{i} + 2xy\vec{j}$

$$\nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 - y^2 & 2xy & 0 \end{vmatrix}$$

$$= \vec{i}[0-0] - \vec{j}[0-0] + \vec{k}[2y+2y]$$

$$\nabla \times \vec{F} = 4y\vec{k}$$



RHS:

$$\iint_S \nabla \times \vec{F} \cdot \hat{n} \, ds = \int_0^b \int_0^a 4y\vec{k} \cdot \vec{k} \, dx \, dy$$

$$= \int_0^b \int_0^a 4y \, dx \, dy$$

$$= 4 \int_0^b y [x]_0^a \, dy$$

$$= 4a \int_0^b y \, dy$$

$$= 4a \left[\frac{y^2}{2} \right]_0^b$$

$$\iint_S \nabla \times \vec{F} \cdot \hat{n} \, ds = 2ab^2 \rightarrow (1)$$

\therefore Given $\vec{F} = (x^2 - y^2)\vec{i} + 2xy\vec{j}$

$$d\vec{r} = dx\vec{i} + dy\vec{j} + dz\vec{k}$$

$$\vec{F} \cdot d\vec{r} = (x^2 - y^2)dx + 2xy \, dy$$





LHS

$$\int_C \vec{F} \cdot d\vec{r} = \int_{AB} + \int_{BC} + \int_{CD} + \int_{DA}$$

Along AB [y=0 ⇒ dy=0]

$$\begin{aligned} \int_{AB} (x^2 - y^2) dx + 2xy dy &= \int_0^a x^2 dx \\ &= \left(\frac{x^3}{3} \right)_0^a \\ &= \frac{a^3}{3} \end{aligned}$$

Along BC [x=a ⇒ dx=0]

$$\begin{aligned} \int_{BC} (x^2 - y^2) dx + 2xy dy &= \int_0^b 2ay dy \\ &= 2a \left(\frac{y^2}{2} \right)_0^b \\ &= ab^2 \end{aligned}$$

Along CD [y=b ⇒ dy=0]

$$\begin{aligned} \int_{CD} (x^2 - y^2) dx + 2xy dy &= \int_a^0 (x^2 - b^2) dx \\ &= \left[\frac{x^3}{3} - b^2 x \right]_a^0 \\ &= 0 - \left(\frac{a^3}{3} - ab^2 \right) \\ &= -\frac{a^3}{3} + ab^2 \end{aligned}$$

Along DA [x=0 ⇒ dx=0]

$$\int_{DA} (x^2 - y^2) dx + 2xy dy = 0$$

$$\begin{aligned} \therefore \int_C \vec{F} \cdot d\vec{r} &= \int_{AB} + \int_{BC} + \int_{CD} + \int_{DA} = \frac{a^3}{3} + ab^2 - \frac{a^3}{3} + ab^2 + 0 \\ &= 2ab^2 \rightarrow (2) \end{aligned}$$

From (1) & (2),

LHS = RHS



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Hence Stoke's theorem is verified.