Greens Theorem in a Plane:

If R is a closed region of the xy plane bounded by a simple closed curve C and if M and N are continuous functions of x and y having continuous derivatives in R then

$$\int_C M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$$

where c is a curve traversed in the anticlockwise direction.

① Evaluate by Greens theorem $\int_C (xy+x^2)\,dx + (x^2+y^2)\,dy$ where C is the square formed by $x=-1$, $x=1$, $y=-1$, $y=1$

Let R be the region enclosed by C

By Greens theorem,

$$\int_C M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$$

Here $M = xy + x^2$ $\quad \frac{\partial M}{\partial y} = x$

$N = x^2+y^2$ $\quad \frac{\partial N}{\partial x} = 2x$

$$\int_C (xy+x^2)\,dx + (x^2+y^2)\,dy = \iint_R (2x-x)\,dx\,dy$$

$$= \int_{-1}^{1}\int_{-1}^{1} x\,dx\,dy = \int_{-1}^{1} \left(\frac{x^2}{2}\right)_{-1}^{1} dy$$

$$= \int_{-1}^{1} \left(\frac{1}{2} - \frac{1}{2}\right) dy$$

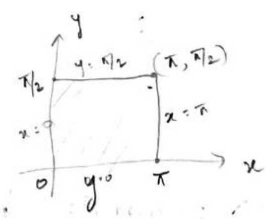
$$= 0.$$

② Evaluate by Greens theorem $\int_C e^{-x}(\sin y\, dx + \cos y\, dy)$ where C is the rectangle with vertices $(0,0), (\pi,0), (\pi, \pi/2), (0,\pi/2)$

Let R be the region enclosed by C

By Greens theorem,

$$\int_C M dx + N dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\, dy.$$



Here $M = e^{-x}\sin y \qquad \frac{\partial M}{\partial y} = e^{-x}\cos y$

$N = e^{-x}\cos y \qquad \frac{\partial N}{\partial x} = -e^{-x}\cos y$

$$\therefore \int_C e^{-x}(\sin y\, dx + \cos y\, dy) = \iint_R (-e^{-x}\cos y - e^{-x}\cos y)\, dx\, dy$$

$$= \int_0^{\pi/2}\int_0^{\pi} (-2e^{-x}\cos y)\, dx\, dy$$

$$= -2\int_0^{\pi/2} \left[-e^{-x}\cos y\right]_0^{\pi} dy$$

$$= 2\int_0^{\pi/2} (e^{-\pi} - 1)\cos y\, dy$$

$$= 2(e^{-\pi}-1)\int_0^{\pi/2} \cos y\, dy$$

$$= 2(e^{-\pi}-1)\left[\sin y\right]_0^{\pi/2}$$

$$= 2(e^{-\pi}-1)$$

3. Evaluate by Green's Theorem $\int_C (x^2 - \cosh y)\,dx + (y + \sin x)\,dy$ where C is the rectangle with vertices $(0,0)$, $(\pi,0)$, $(\pi,1)$ $(0,1)$

Let R be the region enclosed by C

By Green's theorem,

$$\int_C (M\,dx + N\,dy) = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$$

$\frac{d}{dy}(\cosh y) = -\sinh y$

Here $M = x^2 - \cosh y \qquad \frac{\partial M}{\partial y} = \sinh y.$

$N = y + \sin x \qquad \frac{\partial N}{\partial x} = \cos x$

$$\therefore \int_C (x^2 - \cosh y)\,dx + (y + \sin x)\,dy = \iint_R (\cos x - \sinh y)\,dx\,dy$$

$$= \int_0^1 \int_0^\pi (\cos x - \sinh y)\,dx\,dy$$

$$= \int_0^1 \left[\sin x + \cancel{x \sinh y}\right]_0^\pi dy$$

$$= \pi \int_0^1 \sinh y\,dy \quad \pi (\sinh y)_0$$

$$= \pi \left[-\cosh y\right]_0^1 = \pi\left[\cosh 1 - 1\right]$$