



Surface Integral:

Let S be a surface whose projection R_{xy} on the xy plane is \mathcal{D} the points on S have a one-one correspondence with the points on R_{xy} . Let ds be a vector element of the area. then

$$\iint_S \vec{F} \cdot d\vec{s} = \iint_S \vec{F} \cdot \hat{n} ds = \iint_{R_{xy}} \vec{F} \cdot \hat{n} \frac{dx dy}{|\hat{n} \cdot \vec{k}|} \Rightarrow \text{for } xy \text{ plane}$$

$$\iint_S \vec{F} \cdot d\vec{s} = \iint_S \vec{F} \cdot \hat{n} ds = \iint_{R_{yz}} \vec{F} \cdot \hat{n} \frac{dy dz}{|\hat{n} \cdot \vec{j}|} \Rightarrow \text{for } yz \text{ plane}$$

$$\iint_S \vec{F} \cdot \hat{n} ds = \iint_{R_{xz}} \vec{F} \cdot \hat{n} \frac{dx dz}{|\hat{n} \cdot \vec{j}|} \Rightarrow \text{for } xz \text{ plane. } 0 \leq x \leq 0$$

Line Integrals:

Suppose C is an arc and $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$ is the position vector of any point $P(x, y, z)$ on it and \vec{F} is a vector point function at P . Then

$\int_C \vec{F} \cdot d\vec{r}$ is called a line integral of \vec{F} over C .

Line integral $\int_A^B \vec{F} \cdot d\vec{r}$ is also known as the total work done by the force \vec{F} during a displacement from A to B .

① Evaluate $\int_C \vec{F} \cdot d\vec{r}$ where $\vec{F} = x^2y^2\vec{i} + y\vec{j}$ and the curve C

is $y^2 = 4x$ in the xy -plane from $(0,0)$ to $(4,4)$

$$\vec{r} = x\vec{i} + y\vec{j}$$

$$d\vec{r} = dx\vec{i} + dy\vec{j}$$

Given: $\vec{F} = x^2y^2\vec{i} + y\vec{j}$

$$\vec{F} \cdot d\vec{r} = (x^2y^2\vec{i} + y\vec{j}) \cdot (dx\vec{i} + dy\vec{j})$$

$$= x^2y^2 dx + y dy$$

Given: $y^2 = 4x$

$$2y dy = 4 dx$$

$$y dy = 2 dx$$

$$\therefore \vec{F} \cdot d\vec{r} = x^2y^2 dx + 2 dx$$

$$= x^2(4x) dx + 2 dx$$

$$\int_C \vec{F} \cdot d\vec{r} = \int_0^4 (4x^3 + 2) dx$$

$$= \left[\frac{4x^4}{4} + 2x \right]_0^4 = 4^4 + 2(4)$$

$$= 256 + 8 = 264$$

② If $\vec{F} = x^2\vec{i} + xy\vec{j}$ evaluate $\int_C \vec{F} \cdot d\vec{r}$ along the straight

line $y = x$ from $(0,0)$ to $(1,1)$

$$\vec{r} = x\vec{i} + y\vec{j} \Rightarrow d\vec{r} = dx\vec{i} + dy\vec{j}$$

Given: $\vec{F} = x^2\vec{i} + xy\vec{j}$

$$\vec{F} \cdot d\vec{r} = (x^2\vec{i} + xy\vec{j}) \cdot (dx\vec{i} + dy\vec{j})$$

$$= x^2 dx + xy dy$$

Also given: $y = x \Rightarrow dy = dx$.

$$\begin{aligned} \therefore \vec{F} \cdot d\vec{r} &= x^2 dx + x(x)(dx) \\ &= x^2 dx + x^2 dx \\ &= 2x^2 dx \end{aligned}$$

$$\begin{aligned} \int_C \vec{F} \cdot d\vec{r} &= \int_0^1 2x^2 dx \\ &= 2 \int_0^1 x^2 dx = 2 \left[\frac{x^3}{3} \right]_0^1 \end{aligned}$$

$$= \frac{2}{3}$$

③ If $\vec{F} = 5xy\vec{i} + 2y\vec{j}$, evaluate $\int_C \vec{F} \cdot d\vec{r}$ where C is the part of the curve $y = x^3$ between $x=1$ and $x=2$.

$$\vec{F} = 5xy\vec{i} + 2y\vec{j}$$

$$d\vec{r} = dx\vec{i} + dy\vec{j}$$

$$\vec{F} \cdot d\vec{r} = 5xy dx + 2y dy$$

Given: $y = x^3 \Rightarrow dy = 3x^2 dx$

$$\begin{aligned} \int_C \vec{F} \cdot d\vec{r} &= \int_1^2 (5x(x^3) dx + 2(x^3) 3x^2 dx) \\ &= \int_1^2 (5x^4 + 6x^5) dx = \left(\frac{5x^5}{5} + \frac{6x^6}{6} \right) \Big|_1^2 \\ &= [(2)^5 + (2)^6] - [1^5 + 1^6] \\ &= 32 + 64 - 2 \\ &= 94. \end{aligned}$$

④ Find $\int_C \vec{F} \cdot d\vec{r}$ where $\vec{F} = (2y+3)\vec{i} + xz\vec{j} + (yz-x)\vec{k}$ along the line joining the pts $(0,0,0)$ to $(2,1,1)$ Ans: 8

$$\vec{F} \cdot d\vec{r} = (2y+3)dx + xzdy + (yz-x)dz$$

The eqn of the line joining the pts $(0,0,0)$ & $(2,1,1)$ is

SNS COLLEGE OF TECHNOLOGY
(AN AUTONOMOUS INSTITUTION)
COIMBATORE - 35
DEPARTMENT OF MATHEMATICS

$$\frac{x-0}{0-2} = \frac{y-0}{0-1} = \frac{z-0}{0-1} \Rightarrow \frac{x}{2} = \frac{y}{1} = \frac{z}{1} = k \text{ (say)}$$

$$x = 2t, y = t, z = t \Rightarrow dx = 2dt, dy = dt, dz = dt$$

$$\begin{aligned} \vec{F} \cdot d\vec{r} &= (2t+3)(2dt) + 2t^2dt + (t^2-2t)dt \\ &= (4t+6) + 2t^2+t^2-2t) dt = (3t^2+2t+6)dt \end{aligned}$$

$$\text{For } (0,0,0) \Rightarrow t=0 \text{ \& } (2,1,1) \Rightarrow t=1$$

$$\begin{aligned} \int_C \vec{F} \cdot d\vec{r} &= \int_0^1 (3t^2+2t+6) dt = [t^3+t^2+6t]_0^1 \\ &= 1+1+6 = 8 \end{aligned}$$

6. If $\vec{F} = (3x^2+6y)\vec{i} - 14yz\vec{j} + 20xz^2\vec{k}$. Evaluate $\int_C \vec{F} \cdot d\vec{r}$ from $(0,0,0)$ to $(1,1,1)$ along the curve $x=t, y=t^2, z=t^3$.

$$\vec{F} \cdot d\vec{r} = (3x^2+6y)dx - 14yz dy + 20xz^2 dz$$

$$\text{Given: } x=t, y=t^2, z=t^3 \Rightarrow dx=dt, dy=2t dt, dz=3t^2 dt$$

$$\vec{F} \cdot d\vec{r} = (3t^2+6t^2) dt - 14t^5 \cdot 2t dt + 20t \cdot t^6 \cdot 3t^2 dt$$

$$= (9t^2 - 28t^6 + 60t^9) dt$$

$$\int_C \vec{F} \cdot d\vec{r} = \int_0^1 (9t^2 - 28t^6 + 60t^9) dt = \left[\frac{9t^3}{3} - \frac{28t^7}{7} + \frac{60t^{10}}{10} \right]_0^1$$

$$= [3t^3 - 4t^7 + 6t^{10}]_0^1$$

$$= 3 - 4 + 6$$

$$= 5$$