



Stoke's Theorem:

The line integral of the tangential component of a vector function \vec{F} around a simple closed curve C is equal to the surface integral of the normal component of curl \vec{F} over an open surface S .

$$\text{ie., } \int_C \vec{F} \cdot d\vec{r} = \iint_S (\nabla \times \vec{F}) \cdot \hat{n} \, ds$$

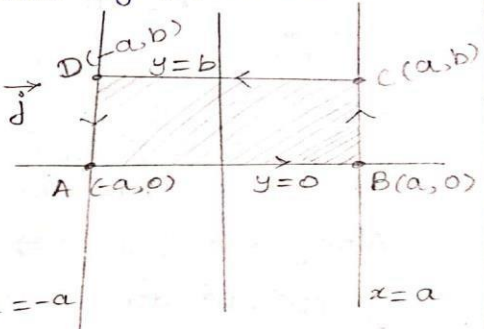
Q. verify Stoke's Theorem for $\vec{F} = (x^2 + y^2)\vec{i} - 2xy\vec{j}$ taken around the rectangle bounded by the lines $x = \pm a, y = 0, y = b$.

Soln.

Given $\vec{F} = (x^2 + y^2)\vec{i} - 2xy\vec{j}$

ST

$$\int_C \vec{F} \cdot d\vec{r} = \iint_S \nabla \times \vec{F} \cdot \hat{n} \, ds$$



Now,

$$\begin{aligned} \nabla \times \vec{F} &= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 + y^2 & -2xy & 0 \end{vmatrix} \\ &= \vec{i} [0 - 0] - \vec{j} [0 - 0] + \vec{k} [-2y - 2y] \\ &= -4y \vec{k} \end{aligned}$$

RHS

$$\begin{aligned} \iint_S \nabla \times \vec{F} \cdot \hat{n} \, ds &= \iint_S (-4y\vec{k}) \cdot \vec{k} \, dx \, dy \\ &= \iint_S (-4y) \, dx \, dy \\ &= -4 \int_0^b \int_{-a}^a y \, dx \, dy \\ &= -4 \int_0^b y [x]_{-a}^a \, dy \end{aligned}$$





$$= -4 \int_0^b y [a+a] dy$$

$$= -8a \int_0^b y dy$$

$$= -8a \left[\frac{y^2}{2} \right]_0^b$$

$$\iint_S (\nabla \times \vec{F}) \cdot \hat{n} ds = -4ab^2 \rightarrow (1)$$

Given $\vec{F} = (x^2 + y^2) \vec{i} - 2xy \vec{j}$

$$d\vec{r} = dx \vec{i} + dy \vec{j} + dz \vec{k}$$

$$\vec{F} \cdot d\vec{r} = (x^2 + y^2) dx - 2xy dy$$

LHS:

$$\int_C \vec{F} \cdot d\vec{r} = \int_{AB} + \int_{BC} + \int_{CD} + \int_{DA}$$

Along AB [$y=0 \Rightarrow dy=0$]

$$\int_{AB} (x^2 + y^2) dx - 2xy dy = \int_{-a}^a x^2 dx$$

$$= \left(\frac{x^3}{3} \right)_{-a}^a$$

$$= \frac{a^3}{3} - \frac{(-a)^3}{3}$$

$$= \frac{2a^3}{3}$$

$$\int_{AB} (x^2 + y^2) dx - 2xy dy = \frac{2a^3}{3}$$

AB

Along BC [$x=a \Rightarrow dx=0$]

$$\int_{BC} (x^2 + y^2) dx - 2xy dy = \int_0^b [0 - 2ay dy]$$

BC

$$= -2a \int_0^b y dy$$





$$= -2a \left[\frac{y^2}{2} \right]_0^b$$

$$\int_{BC} (x^2 + y^2) dx - 2xy dy = -ab^2$$

Along CD $[-y=b \Rightarrow dy=0]$

$$\int_{CD} (x^2 + y^2) dx - 2xy dy = \int_a^{-a} (x^2 + b^2) dx$$

$$= \left[\frac{x^3}{3} + b^2 x \right]_a^{-a}$$

$$= \left(\frac{-a^3}{3} - ab^2 \right) - \left(\frac{a^3}{3} + ab^2 \right)$$

$$= -2ab^2 - \frac{2a^3}{3}$$

Along DA $(x=-a \Rightarrow dx=0)$

$$\int_{DA} (x^2 + y^2) dx - 2xy dy = \int_b^0 0 - 2(-a)y dy$$

$$= \int_b^0 2ay dy$$

$$= 2a \left[\frac{y^2}{2} \right]_b^0$$

$$= 0 - ab^2$$

$$= -ab^2$$

$$\therefore \int_C \vec{F} \cdot d\vec{r} = \int_{AB} + \int_{BC} + \int_{CD} + \int_{DA}$$

$$= \frac{2a^3}{3} - ab^2 - 2ab^2 - \frac{2a^3}{3} - ab^2$$

$$= -4ab^2 \rightarrow (2)$$

$$\text{LHS} = \text{RHS}$$

From (1) & (2),

Hence Stoke's theorem is verified.

