



Green's Theorem:

If $M, N, \frac{\partial M}{\partial y}, \frac{\partial N}{\partial x}$ are continuous and one-valued functions in a region R enclosed by the curve C , then

$$\int_C [Mdx + Ndy] = \iint_R \left[\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right] dx dy$$

Problems:

1. Evaluate $\int_C (x^2 + xy) dx + (x^2 + y^2) dy$, where C is the square bounded by the lines $x=0, x=1, y=0$ and $y=1$.

Soln.

Green's Theorem:

$$\int_C [Mdx + Ndy] = \iint_R \left[\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right] dx dy$$

$$\text{Here } M = xy + x^2 \quad \left| \quad N = x^2 + y^2 \right.$$

$$\frac{\partial M}{\partial y} = x \quad \left| \quad \frac{\partial N}{\partial x} = 2x \right.$$

$$\text{Now, } \int_C [Mdx + Ndy] = \iint_R [2x - x] dx dy$$

$$= \int_0^1 \int_0^1 x dx dy = \int_0^1 \left[\frac{x^2}{2} \right]_0^1 dy$$

$$= \int_0^1 \left[\frac{1}{2} - 0 \right] dy$$

$$= \frac{1}{2} [y]_0^1$$

$$\int_C [(x^2 + xy) dx + (x^2 + y^2) dy] = \frac{1}{2}$$





Q. Verify Green's theorem for $\int_C (xy + y^2) dx + x^2 dy$ where C is the closed curve bounded by $y = x^2$ and $y = x$.

Soln.

By Green's theorem,

$$\int_C [M dx + N dy] = \iint_R \left[\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right] dx dy$$

Given $y = x^2$; $y = x$

$$\Rightarrow x^2 = x$$

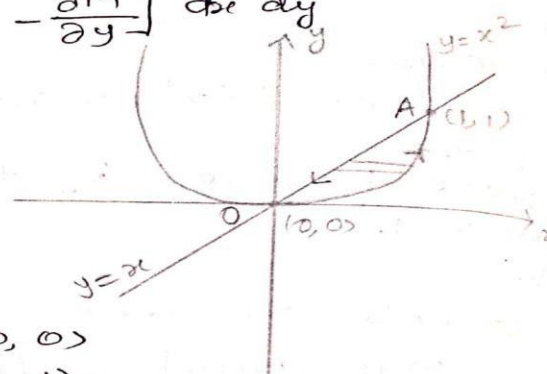
$$x^2 - x = 0$$

$$x(x-1) = 0$$

$$x = 0, x = 1$$

when $x = 0, y = 0 \Rightarrow (0, 0)$

$x = 1, y = 1 \Rightarrow (1, 1)$



Here $M = xy + y^2$

$$\frac{\partial M}{\partial y} = x + 2y$$

$N = x^2$

$$\frac{\partial N}{\partial x} = 2x$$

RHS

$$\iint_R \left[\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right] dx dy = \int_0^1 \int_y^{\sqrt{y}} [2x - (x + 2y)] dx dy$$

$$= \int_0^1 \int_y^{\sqrt{y}} [x - 2y] dx dy$$

$$= \int_0^1 \left[\frac{x^2}{2} - 2xy \right]_{x=y}^{\sqrt{y}} dy$$

$$= \int_0^1 \left[\left(\frac{y}{2} - 2y^{3/2} \right) - \left(\frac{y^2}{2} - 2y^2 \right) \right] dy$$

$$= \int_0^1 \left[\frac{y}{2} - 2y^{3/2} + \frac{3}{2}y^2 \right] dy$$





$$= \int_0^1 \left[\frac{y^2}{4} - 2 \frac{y^{5/2}}{5/2} + \frac{3}{2} \frac{y^3}{3} \right]_0^1$$

$$= \left(\frac{1}{4} - \frac{4}{5} + \frac{1}{2} \right) - 0$$

$$= \frac{5 - 16 + 10}{20}$$

RHS = $\frac{-1}{20}$

LHS

To evaluate $\int [M dx + N dy]$, we shall take C in the different paths.

i). along AO [$y = x$]

ii). along OA [$y = x^2$]

Along with AO [$y = x \Rightarrow dy = dx$]

$$\int_{AO} [M dx + N dy] = \int_{AO} [(xy + y^2) dx + x^2 dy]$$

$$= \int_0^1 [(x^2 + x^2) dx + x^2 dx]$$

$$= \int_0^1 [x^2 + x^2 + x^2] dx$$

$$= 3 \int_0^1 x^2 dx$$

$$= 3 \left[\frac{x^3}{3} \right]_0^1$$

$$= [0 - 1]$$

$$= -1$$

Along with OA [$y = x^2 \Rightarrow dy = 2x dx$]

$$\int_{OA} [(xy + y^2) dx + x^2 dy] = \int_0^1 [(x(x^2) + x^4) dx + x^2 (2x dx)]$$



$$\begin{aligned}
 &= \int_0^1 [x^3 + x^4 + 2x^3] dx \\
 &= \left[\frac{x^4}{4} + \frac{x^5}{5} + \frac{2x^4}{4} \right]_0^1 \\
 &= \left(\frac{1}{4} + \frac{1}{5} + \frac{1}{2} \right) - 0 \\
 &= \frac{5+4+10}{20} \\
 &= \frac{19}{20}
 \end{aligned}$$

$$\begin{aligned}
 \therefore \int_C (M dx + N dy) &= \int_{OA} + \int_{AO} = \frac{19}{20} - 1 \\
 &= \frac{19-20}{20}
 \end{aligned}$$

$$\int_C (M dx + N dy) = \frac{-1}{20}$$

\therefore LHS = RHS

Hence green's theorem is verified.

3]. Verify green's theorem for

$\int_C (x^2 - y^2) dx + 2xy dy$ where C is the closed curve bounded by $y = x^2$ and $y^2 = x$

Soln.

Given $y = x^2$ and $y^2 = x$

$$\Rightarrow y = (y^2)^2$$

$$y^4 - y = 0$$

$$y(y^3 - 1) = 0$$

$$y = 0, \quad y^3 - 1 = 0$$

$$y^3 = 1 \Rightarrow y = 1$$

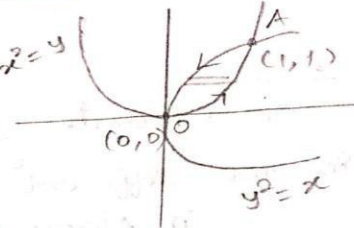




UNIT-1 VECTOR CALCULUS

GREEN'S THEOREM

when $y=0 \Rightarrow x=0 \Rightarrow (0,0)$
 $y=1 \Rightarrow x=1 \Rightarrow (1,1)$



By Green's Theorem,

$$\int_C M dx + N dy = \iint_R \left[\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right] dx dy$$

Here $M = x^2 - y^2$ $N = 2xy$
 $\frac{\partial M}{\partial y} = -2y$ $\frac{\partial N}{\partial x} = 2y$

RHS

$$\iint_R \left[\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right] dx dy = \int_0^1 \int_{y^2}^{\sqrt{y}} [2y + 2y] dx dy$$

$$= \int_0^1 4y(x) \Big|_{y^2}^{\sqrt{y}} dy$$

$$= \int_0^1 [4y y^{1/2} - 4y y^2] dy = \int_0^1 [4y^{3/2} - 4y^3] dy$$

$$= 4 \left[\frac{y^{3/2+1}}{3/2+1} - \frac{y^4}{4} \right]_0^1$$

$$= 4 \left[\frac{2}{5} y^{5/2} - \frac{y^4}{4} \right]_0^1 = 4 \left[\frac{2}{5} - \frac{1}{4} \right]$$

$$= 4 \left[\frac{8-5}{20} \right]$$

$$= \frac{12}{20}$$

$$\iint_R \left[\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right] dx dy = \frac{3}{5}$$





ΔHS

To evaluate $\int [M dx + N dy]$, we shall take C in the different paths.

i). Along OA $[y = x^2]$

ii). Along AO $[y^2 = x]$

Along OA $[y = x^2 \Rightarrow dy = 2x dx]$

$$\begin{aligned} \int_{OA} [(x^2 - y^2) dx + 2xy dy] &= \int_0^1 [(x^2 - x^4) dx + 2x(x^2)(2x dx)] \\ &= \int_0^1 [x^2 - x^4 + 4x^4] dx \\ &= \int_0^1 [3x^4 + x^2] dx = \left[\frac{3x^5}{5} + \frac{x^3}{3} \right]_0^1 \\ &= \left(\frac{3}{5} + \frac{1}{3} \right) - 0 \\ &= \frac{9+5}{15} - 0 \\ &= \frac{14}{15} \end{aligned}$$

Along AO $[y^2 = x \Rightarrow 2y dy = dx \Rightarrow dy = \frac{dx}{2\sqrt{x}}$

$$\begin{aligned} \int_{AO} [(x^2 - y^2) dx + 2xy dy] &= \int_1^0 [(x^2 - x) dx + 2x x^{1/2} \frac{dx}{2\sqrt{x}}] \\ &= \int_1^0 [x^2 - x + x] dx = \int_1^0 x^2 dx \\ &= \left[\frac{x^3}{3} \right]_1^0 = 0 - \frac{1}{3} = -\frac{1}{3} \end{aligned}$$





Now,

$$\int_C (x^2 - y^2) dx + 2xy dy = \int_{OA} + \int_{AO}$$
$$= \frac{14}{15} - \frac{1}{3}$$
$$= \frac{14 - 5}{15}$$
$$= \frac{9}{15}$$
$$= \frac{3}{5}$$

\therefore LHS = RHS
Hence verified.