Legendre's linear differential Equation

An eqn. of the form

$$(ax+b)^n \frac{d^n y}{dx^n} + a_1 (ax+b)^{n-1} \frac{d^{n-1}y}{dx^{n-1}} + a^2(ax+b)^{n-2} \frac{d^{n-2}y}{dx^{n-2}}$$

$$+ \cdots + a_{n-1}(ax+b)\frac{dy}{dx} + a_n y = Q(x) \rightarrow (1)$$

Take $ax+b = e^z$

$z = \log(ax+b)$

$(ax+b)D = aD'$

$(ax+b)^2 D^2 = a^2 D'(D'-1)$

$(ax+b)^3 D^3 = a^3 D'(D'-1)(D'-2)$ and so on.

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1. Transform the equation to constant coefficients eqn.

$(2x+3)^2 y'' - (2x+3)y' + 2y = 6x$

Soln.

Given $[(2x+3)^2 D^2 - (2x+3)D + 2]y = 6x$ $\rightarrow (1)$

Take $2x+3 = e^z \Rightarrow 2x = e^z - 3$

$z = \log(2x+3)$ $\Rightarrow x = \dfrac{e^z - 3}{2}$

$(2x+3)D = 2D'$

$(2x+3)^2 D^2 = 4D'(D'-1)$

$(1) \Rightarrow [4D'(D'-1) - 2D' + 2]y = 6\left[\dfrac{e^z-3}{2}\right]$

$[4D'^2 - 4D' - 2D' + 2]y = 3[e^z - 3]$

$[4D'^2 - 6D' + 2]y = 3e^z - 9$ which is a linear eqn. with constant coefficients.

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2. Solve $(x+2)^2 \dfrac{d^2y}{dx^2} - (x+2)\dfrac{dy}{dx} + y = 3x + 4$

Soln.

Given $[(x+2)^2 D^2 - (x+2) D + 1] y = 3x + 4 \longrightarrow (1)$

Take $x + 2 = e^z \Rightarrow x = e^z - 2$

$z = \log(x+2)$

$(x+2) D = D'$

$(x+2)^2 D^2 = D'(D'-1)$

$(1) \Rightarrow [D'(D'-1) - D' + 1] y = 3[e^z - 2] + 4$

$[D'^2 - D' - D' + 1] y = 3e^z - 6 + 4$

$[D'^2 - 2D' + 1] y = 3e^z - 2$

CF

$m^2 - 2m + 1 = 0$

$(m-1)(m-1) = 0$

$m = 1, 1$

$CF = (A + Bz) e^z$

$PI_1 = \frac{1}{D'^2 - 2D' + 1} 3e^z + \frac{1}{D'^2 - 2D' + 1} 2e^{0z}$

$= \frac{1}{1 - 2 + 1} 3e^z - 2 \frac{1}{1} e^{0z}$ $\quad D' \to a = 1$, $a = 0$

$= z \frac{1}{2D' - 2} 3e^z - 2$

$= z \frac{1}{2(1) - 2} 3e^z - 2$

$= z^2 \frac{1}{2} 3e^z - 2$

$PI = \frac{3z^2 e^z}{2} - 2$

The Soln. is, $y = CF + PI$

$= (A + Bz) e^z + \frac{3z^2 e^z}{2} - 2$

$y = [A + B \log(x+2)](x+2) + \frac{3[\log(x+2)]^2 (x+2)}{2} - 2$