UNIT-II ORDINARY DIFFERENTIAL EQUATIONS   Legendre's Linear Differential Equation

2]. Solve $(3x+2)^2 \dfrac{d^2y}{dx^2} + 3(3x+2)\dfrac{dy}{dx} - 36y = 3x^2 + 4x + 1$

Soln.

Given $\left[(3x+2)^2 D^2 + 3(3x+2)D - 36\right] y = 3x^2 + 4x + 1$

Take $3x+2 = e^z \Rightarrow 3x = e^z - 2 \Rightarrow x = \dfrac{e^z - 2}{3} \hookrightarrow (1)$

$\log(3x+2) = z$

$(3x+2)D = 3D'$

$(3x+2)^2 D^2 = 9D'(D'-1)$

$(1) \Rightarrow \left[9D'(D'-1) + 3(3D') - 36\right] y = 3\left[\dfrac{e^z - 2}{3}\right]^2 + 4\left[\dfrac{e^z - 2}{3}\right] + 1$

$\left[9D'^2 - 9D' + 9D' - 36\right] y = \dfrac{3}{9}\left[e^{2z} + 4 - 4e^z\right] + \dfrac{4}{3}e^z - \dfrac{8}{3} + 1$

$\left[9D'^2 - 36\right] y = \dfrac{e^{2z}}{3} + \dfrac{4}{3} - \dfrac{4e^z}{3} + \dfrac{4}{3}e^z - \dfrac{8}{3} + 1$

$= \dfrac{e^{2z}}{3} - \dfrac{1}{3}$

$9(D'^2 - 4) y = \dfrac{1}{3}\left[e^{2z} - 1\right]$

$(D'^2 - 4) y = \dfrac{1}{27}\left[e^{2z} - 1\right]$

CF: $m^2 - 4 = 0 \Rightarrow m^2 = 4$

$m = \pm 2$

$CF = A e^{2z} + B e^{-2z}$

$PI = \dfrac{1}{D'^2 - 4}\left[\dfrac{1}{27}(e^{2z} - 1)\right]$

$= \dfrac{1}{27}\left[\dfrac{1}{D'^2 - 4} e^{2z} - \dfrac{1}{D'^2 - 4} e^{0z}\right]$

$= \dfrac{1}{27}\left[z \dfrac{1}{2D'} e^{2z} - \dfrac{1}{(-4)} e^{0z}\right] \qquad 1 \to D' \to 2$
$\qquad\qquad\qquad\qquad\qquad\qquad 2 \to D' \to 0$

$= \dfrac{1}{27}\left[\dfrac{z}{4} e^{2z} + \dfrac{1}{4}\right]$

$= \dfrac{1}{108}\left[z e^{2z} + 1\right]$

The Soln. is, $y = CF + PI = A e^{2z} + B e^{-2z} + \dfrac{1}{108}\left[z e^{2z} + 1\right]$

$y = A(3x+2)^2 + B(3x+2)^{-2} + \dfrac{1}{108}\left[\log(3x+2)[3x+2]^2 + 1\right]$