Solve $(D^2+a^2)y = \sec ax$ using method of variation of parameters

Given: $(D^2+a^2)y = \sec ax$

$m^2+a^2=0 \Rightarrow m^2=a^2$

$\Rightarrow m = \pm ai$

$CF = A\cos ax + B\sin ax$

Here $f_1 = \cos ax \qquad f_2 = \sin ax$

$f_1' = -a\sin ax \qquad f_2' = a\cos ax$

$W = f_1 f_2' - f_1' f_2$

$= \cos ax(a\cos ax) + (a\sin ax)\sin ax$

$= a\cos^2 ax + a\sin^2 ax$

$= a[\cos^2 ax + \sin^2 ax] = a$

$\Rightarrow W = a$

$PI = Pf_1 + Qf_2$

$$P = -\int \frac{f_2 X}{W}dx = -\int \frac{\sin ax \sec ax}{a}dx$$

$$= \frac{-1}{a}\int \sin ax \frac{1}{\cos ax}dx = \frac{-1}{a}\int \tan ax\, dx$$

$$= \frac{-1}{a} \frac{\log(\sec ax)}{a} = \frac{-1}{a^2} \log(\sec ax)$$

$$Q = \int \frac{f_1 x}{\omega} dx = \int \frac{\cos ax \sec ax}{a} dx$$

$$= \frac{1}{a} \int \cos ax \frac{1}{\cos ax} dx = \frac{1}{a} \int dx$$

$$Q = \frac{x}{a}$$

$$PI = \frac{-1}{a^2} \log(\sec ax) \cos ax + \frac{x}{a} \sin ax$$

The general solution is,

$$y = CF + PI$$

$$= A\cos ax + B\sin ax - \frac{1}{a^2} \log(\sec ax)\cos ax + \frac{x}{a} \sin ax$$