PART A

1. If $\phi = x^2 + y^2 + z^2$, find $\nabla\phi$ at (1,1,-1)

Given, $\phi = x^2 + y^2 + z^2$ -------------------- **(i)**

There fore $\nabla\phi = \vec{i}\frac{\partial\phi}{\partial x} + \vec{j}\frac{\partial\phi}{\partial y} + \vec{k}\frac{\partial\phi}{\partial z}$ --------- **(ii)**

From (i), $\frac{\partial\phi}{\partial x} = 2x$; $\frac{\partial\phi}{\partial y} = 2y$; $\frac{\partial\phi}{\partial z} = 2z$ --------------**(iii)**

Sub (iii) in (i) , we get

$$\nabla\phi = \vec{i}(2x) + \vec{j}(2y) + \vec{k}(2z)$$

There fore, $(\nabla\phi)_{at(1,1,-1)} = 2\vec{i} + 2\vec{j} - 2\vec{k}$

2. Find grad r^n, where $r = |\vec{r}|$ and $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$

Given, $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$

$$|\vec{r}| = r = \sqrt{x^2 + y^2 + z^2}$$

$r^2 = x^2 + y^2 + z^2$------------------ **(i)**

Diff (i) partially w.r.t 'x'

$$2r\frac{\partial r}{\partial x} = 2x$$

$$\Rightarrow \frac{\partial r}{\partial x} = \frac{x}{r}$$

$$\frac{\partial r}{\partial y} = \frac{y}{r} \text{ and } \frac{\partial r}{\partial z} = \frac{z}{r}$$

$$\therefore \text{grad} r^n = \nabla r^n$$

$$= \sum \vec{i}\frac{\partial}{\partial x}(r^n)$$

$$= \sum \vec{i}\frac{\partial}{\partial x}(r^n).\frac{\partial r}{\partial x}$$

$$= \vec{i} n.r^{n-1}\frac{x}{r}$$

$$= \vec{i} n.r^{n-2}.x$$

$$= n.r^{n-2}\left[x\vec{i} + y\vec{j} + z\vec{k}\right]$$

$$= n.r^{n-2}\vec{r}$$

3. Find the unit vector normal to the surface $x^2+y^2-z=10$ **at (1,1,1).**

Given $\phi = x^2+y^2-z=10$

$$\nabla\phi = \vec{i}\frac{\partial\phi}{\partial x}+\vec{j}\frac{\partial\phi}{\partial y}+\vec{k}\frac{\partial\phi}{\partial z}$$

$$= 2x\vec{i}+2y\vec{j}-\vec{k}$$

$$\therefore (\nabla\phi)_{at(1,1,1)} = 2\vec{i}+2\vec{j}-\vec{k}$$

$$|\nabla\phi| = \sqrt{4+4+1} = \sqrt{9} = 3$$

Unit normal vector $\hat{n} = \frac{\nabla\phi}{|\nabla\phi|} = \frac{2\vec{i}+2\vec{j}-\vec{k}}{3}$

4. Find the directional derivative of $\phi = xy+yz+xz$ **at the point (1,2,3) in the direction** $3\vec{i}+4\vec{j}+5\vec{k}$**.**

Given, $\phi = xy+yz+xz$ **------------------------(i)**

Let $\vec{n} = 3\vec{i}+4\vec{j}+5\vec{k}$ **----------------------- (ii)**

Directional derivative = $(\nabla\phi).\hat{n}$ **--------- (A)**

$$\nabla\phi = \vec{i}\frac{\partial\phi}{\partial x}+\vec{j}\frac{\partial\phi}{\partial y}+\vec{k}\frac{\partial\phi}{\partial z}$$

From (i), $\frac{\partial\phi}{\partial x} = y+z$ **;** $\frac{\partial\phi}{\partial y} = x+z$ **;** $\frac{\partial\phi}{\partial z} = y+x$

$$\therefore \nabla\phi = (y+z)\vec{i}+(x+z)\vec{j}+(y+x)\vec{k}$$

$$\therefore (\nabla\phi)_{at(1,2,3)} = 5\vec{i}+4\vec{j}+3\vec{k}$$

From (ii), we have

$$\hat{n} = \frac{\vec{n}}{|\vec{n}|} = \frac{3\vec{i}+4\vec{j}+5\vec{k}}{\sqrt{50}}$$ **------------------- (iv)**

Sub (iii) and (iv) in (A), we get

Directional derivative = $(\nabla\phi).\hat{n} = (5\vec{i}+4\vec{j}+3\vec{k}).\frac{3\vec{i}+4\vec{j}+5\vec{k}}{\sqrt{50}}$

$$= \frac{15+16+15}{\sqrt{25x2}} = \frac{46}{5\sqrt{2}}$$

5. In what direction from the point (1,-1,-2) is the directional derivative of $\phi = x^3y^3z^3$ a maximum? What is the magnitude of this maximum?

Given, $\phi = x^3y^3z^3$ ---------------------(i)

$$\nabla\phi = \vec{i}\frac{\partial\phi}{\partial x} + \vec{j}\frac{\partial\phi}{\partial y} + \vec{k}\frac{\partial\phi}{\partial z}$$

From (i), $\frac{\partial\phi}{\partial x} = 3x^2y^3z^3$; $\frac{\partial\phi}{\partial y} = 3x^3y^2z^3$; $\frac{\partial\phi}{\partial z} = 3x^3y^3z^2$

$$\therefore \nabla\phi = 3x^2y^3z^3\vec{i} + 3x^3y^2z^3\vec{j} + 3x^3y^3z^2\vec{k}$$

$$\therefore (\nabla\phi)_{at(1,2,3)} = 24\vec{i} - 24\vec{j} - 12\vec{k}$$

There fore the directional derivative is maximum in the direction $24\vec{i} - 24\vec{j} - 12\vec{k}$.

Magnitude of this maximum is $|\nabla\phi|$

$$= \sqrt{(24)^2 + (-24)^2 + (-12)^2}$$

$$= \sqrt{1296} = 36$$

6. Find the angle between the normal to the surface $xy = z^2$ at the points (1,4,2) and (-3,-3,3).

Let $\phi = xy - z^2$ --------------------(i)

$$\therefore \nabla\phi = y\vec{i} + x\vec{j} - 2z\vec{k}$$

Normal to the surface is $\nabla_1\phi$ **and** $\nabla_2\phi$

$$\therefore \nabla_1\phi = (\nabla\phi)_{at(1,4,2)} = 4\vec{i} + \vec{j} - 4\vec{k}$$

$$\nabla_2\phi = (\nabla\phi)_{at(-3,-3,3)} = -3\vec{i} - 3\vec{j} - 6\vec{k}$$

$$\therefore |\nabla_1\phi| = \sqrt{33}; |\nabla_2\phi| = \sqrt{54}$$

There fore angle between the normal to the surface is,

$$\cos\theta = \frac{(\nabla_1\phi)(\nabla_2\phi)}{|\nabla_1\phi||\nabla_2\phi|} = \frac{(4\vec{i} + \vec{j} - 4\vec{k}).(-3\vec{i} - 3\vec{j} - 6\vec{k})}{\sqrt{33}\sqrt{54}}$$

$$= \frac{9}{\sqrt{1782}} = \frac{9}{9\sqrt{22}} = \frac{1}{\sqrt{22}}$$

$$\therefore \quad \theta = \cos^{-1}\left[\frac{1}{\sqrt{22}}\right]$$

7. If ϕ is a scalar point function, then prove that curl (grad ϕ)=0.

$$\text{grad}\,\phi = \vec{i}\frac{\partial\phi}{\partial x}+\vec{j}\frac{\partial\phi}{\partial y}+\vec{k}\frac{\partial\phi}{\partial z}$$

$$\text{curl grad}\,\phi = \nabla \times \left[\vec{i}\frac{\partial\phi}{\partial x}+\vec{j}\frac{\partial\phi}{\partial y}+\vec{k}\frac{\partial\phi}{\partial z}\right]$$

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial\phi}{\partial x} & \frac{\partial\phi}{\partial x} & \frac{\partial\phi}{\partial x} \end{vmatrix}$$

$$= \vec{i}\left[\frac{\partial^2\phi}{\partial y\partial z}-\frac{\partial^2\phi}{\partial y\partial z}\right]-\vec{j}\left[\frac{\partial^2\phi}{\partial x\partial z}-\frac{\partial^2\phi}{\partial x\partial z}\right]+\vec{k}\left[\frac{\partial^2\phi}{\partial x\partial y}-\frac{\partial^2\phi}{\partial x\partial y}\right]$$

$$= 0$$

8. If $\vec{A}$ is a constant vector, prove that div $\vec{A}$ = 0.

Let $\vec{A} = A_1\vec{i}+A_2\vec{j}+A_3\vec{k}$

Where A_1, A_2, A_3 **are constants**

$\therefore$ **div** $\vec{A} = \nabla\cdot\vec{A}$

$$= \left(\vec{i}\frac{\partial}{\partial x}+\vec{j}\frac{\partial}{\partial y}+\vec{k}\frac{\partial}{\partial z}\right).(A_1\vec{i}+A_2\vec{j}+A_3\vec{k})$$

$$= \frac{\partial A_1}{\partial x}+\frac{\partial A_2}{\partial y}+\frac{\partial A_3}{\partial z} = 0+0+0$$

div $\vec{A} = 0$

9. If $\vec{A}$ is a constant vector, prove that curl $\vec{A}$ = 0.

Let $\vec{A} = A_1\vec{i}+A_2\vec{j}+A_3\vec{k}$

Where A_1, A_2, A_3 **are constants**

$$\text{curl}\,\vec{A} = \nabla\times\vec{A} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_1 & A_2 & A_3 \end{vmatrix}$$

$$= \vec{i}\left[\frac{\partial A_3}{\partial y}-\frac{\partial A_2}{\partial z}\right]-\vec{j}\left[\frac{\partial A_3}{\partial x}-\frac{\partial A_1}{\partial z}\right]+\vec{k}\left[\frac{\partial A_2}{\partial x}-\frac{\partial A_1}{\partial y}\right]$$

$$= \vec{i}(0-0)-\vec{j}(0-0)+\vec{k}(0-0)$$

curl $\vec{A} = 0$

10. Determine f(r) so that the vector f(r)$\vec{r}$ is solenoidal.

Since $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$

f(r) = $xf(r)\vec{i} + yf(r)\vec{j} + zf(r)\vec{k}$

div [f(r)] = $\frac{\partial}{\partial x}[xf(r)] + \frac{\partial}{\partial y}[yf(r)] + \frac{\partial}{\partial z}[zf(r)]$

$$= f(r) + xf'(r)\frac{\partial r}{\partial x} + yf'(r)\frac{\partial r}{\partial y} + f(r) + f(r) + zf'(r)\frac{\partial r}{\partial z}$$

$$= 3f(r) + f'(r)\left[x\frac{\partial r}{\partial x} + y\frac{\partial r}{\partial y} + z\frac{\partial r}{\partial z}\right]$$

$$= 3f(r) + f'(r)\left[x\frac{x}{r} + y\frac{y}{r} + z\frac{z}{r}\right]$$

$$= 3f(r) + \frac{f'(r)}{r}\left[x^2 + y^2 + z^2\right]$$

$$= 3f(r) + rf'(r)$$

Since f(r)$\vec{r}$ is solenoidal, div[f(r)$\vec{r}$] = 0

ie., $3f(r) + rf'(r)$ **= 0**

$$\frac{f'(r)}{f(r)} = \frac{-3}{r}$$

Integrating w.r.t r, we get

log f(r) = -3log r + log c

log f(r) = log cr^{-3}

f(r) = cr^{-3}

f(r) = $\frac{c}{r^3}$

11.Find the value of 'a' so that the vector, $\vec{F} = (x+3y)\vec{i} + (y-2z)\vec{j} + (x+az)\vec{k}$ **is Solenoidal.**

Given $\vec{F}$ **is solenoidal.**

div $\vec{F}$ **= 0**

ie., $\nabla \cdot \vec{F}$ **= 0**

ie., $\left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right) \cdot \left[(x+3y)\vec{i} + (y-2z)\vec{j} + (x+az)\vec{k}\right]$ **= 0**

ie., $\frac{\partial}{\partial x}(x+3y) + \frac{\partial}{\partial y}(y-2z) + \frac{\partial}{\partial z}(x+az) = 0 \Rightarrow 1+1+a=0 \Rightarrow a=-2$

12.Show that the vector $2xy\vec{i} + (x^2+2yz)\vec{j} + (y^2+1)\vec{k}$ **is irrotational.**

Let $\vec{F} = 2xy\vec{i} + (x^2+2yz)\vec{j} + (y^2+1)\vec{k}$

A vector $\vec{F}$ **is said to be irrotational if** $\nabla \times \vec{F}$ **= 0**

Now, $\nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2xy & (x^2+2yz) & (y^2+1) \end{vmatrix}$

$$= \vec{i}\left[\frac{\partial(y^2+1)}{\partial y} - \frac{\partial(x^2+2yz)}{\partial z}\right] - \vec{j}\left[\frac{\partial(y^2+1)}{\partial x} - \frac{\partial(2xy)}{\partial z}\right] + \vec{k}\left[\frac{\partial(x^2+2yz)}{\partial x} - \frac{\partial(2xy)}{\partial y}\right]$$

$$= \vec{i}(2y-2y) - \vec{j}(0-0) + \vec{k}(2x-2x)$$

$$\nabla \times \vec{F} = 0$$

13.Show that the vector $\vec{F} = 3y^4z^2\vec{i} + 4x^3z^2\vec{j} - 3x^2y^2\vec{k}$ **is solenoidal.**

We know that, if $\vec{F}$ **is solenoidal, we have**

$$\text{div } \vec{F} = \nabla\cdot\vec{F}$$

$$= \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right)\cdot\left[3y^4z^2\vec{i} + 4x^3z^2\vec{j} - 3x^2y^2\vec{k}\right]$$

$$= \frac{\partial}{\partial x}(3y^4z^2) + \frac{\partial}{\partial y}(4x^3z^2) + \frac{\partial}{\partial z}(-3x^2y^2)$$

$$= 0+0+0$$

$$\therefore \text{div}\vec{F} = 0$$

Hence $\vec{F}$ **is solenoidal.**

14.Define the line integral.

Let $\vec{F}$ **be a vector field in space and let AB be a curve described in the sense A to B. Divide the curve AB into n elements** $\vec{dr_1}, \vec{dr_2}, \ldots\ldots\ldots, \vec{dr_n}$ **.**

Let $\vec{F_1}, \vec{F_2}, \ldots\ldots\ldots, \vec{F_n}$ **be the values of this vector at the junction points of the vectors** $\vec{dr_1}, \vec{dr_2}, \ldots\ldots\ldots, \vec{dr_n}$ **, then the sum**

$$\lim_{n\to\infty} \sum_A^B \vec{F_n}\cdot\vec{dr_n} = \int_A^B \vec{F}\cdot\vec{dr}$$ **is called the line integral.**

If the line integral is along the curve c then it is denoted by

$\int_c \vec{F}\cdot\vec{dr}$ **or** $\oint_c \vec{F}\cdot\vec{dr}$ **if c is a closed curve.**

15. Evaluate $\int_c \vec{F}.d\vec{r}$ along the curve c in xy plane, $y = x^3$ from the point (1,1) to (2,8) if $\vec{F} = (5xy - 6x^2)\vec{i} + (2y - 4x)\vec{j}$.

Given $\vec{F} = (5xy - 6x^2)\vec{i} + (2y - 4x)\vec{j}$, $y = x^3$

Now, $\vec{r} = x\vec{i} + y\vec{j}$; $d\vec{r} = dx\vec{i} + dy\vec{j}$

Here $y = x^3$; $dy = 3x^2dx$

$$\therefore \int_c \vec{F}.d\vec{r} = \int_c \left[(5xy - 6x^2)\vec{i} + (2y - 4x)\vec{j}\right].\left[dx\vec{i} + dy\vec{j}\right]$$

$$= \int_c \left[(5xy - 6x^2)dx + (2y - 4x)dy\right]$$

$$= \int_c \left[(5x(x^3) - 6x^2)dx + [(2x^3 - 4x)3x^2dx\right]$$

$$= \int_c (5x^4 - 6x^2 + 6x^5 - 12x^3)dx$$

$$= x^5 - 2x^3 + x^6 - 3x^4$$

There fore $\int_c \vec{F}.d\vec{r}$ **from the point (1,1) to (2,8)**

ie., $\int_1^2 \vec{F}.d\vec{r} = \left[x^5 - 2x^3 + x^6 - 3x^4\right]_1^2 = 35$

16. Define surface integral.

An integral which is evaluated over a surface is called a surface integral.

$\therefore \lim_{n\to\infty} \sum_{i=1}^{n} \vec{F}(x_i, y_i, z_i).\hat{n}_i \Delta S_i$ **is known as the surface integral.**

17. Find $\iint_s \vec{r}.\vec{ds}$, where s is the surface of the tetrahedron whose vertices are (0,0,0), (1,0,0), (0,1,0), (0,0,1).

By Gauss divergence theorem,

$$\iint_s \vec{r}.\vec{ds} = \iiint_v (\nabla.\vec{r})dv$$

$$\nabla.\vec{r} = \left(\vec{i}\frac{\partial}{\partial x}+\vec{j}\frac{\partial}{\partial y}+\vec{k}\frac{\partial}{\partial z}\right).\left[x\vec{i}+y\vec{j}+z\vec{k}\right] = 1+1+1 = 3$$

$$\therefore \iint_S \vec{r}.\vec{ds} = \iiint_V 3dv = 3v$$

18. If $\vec{F} = \text{curl}\vec{A}$, prove $\iint_S \vec{F}.\hat{n}ds = 0$, for any closed surface S.

By Gauss divergence theorem,

$$\iint_S \vec{F}.\hat{n}\,ds = \iiint_V \nabla.\vec{F}\,dV = \iiint_V div(\vec{F})dv$$

$$= \iiint_V div(curl\,\vec{A})dv = 0 \quad \text{[since div(curl } \vec{A})=0]$$

19. Define Volume integral.

An integral which can be evaluated over a volume closed by a surface is called a volume integral. Volume integral can be evaluated by triple integration.

ie., $\iiint_v f(x,y,z)dv$

20. State Gauss Divergence theorem.

If $\vec{F}$ is a vector point function, finite and differentiable in a region r bounded by a closed surface S, then the surface integral of the normal component of $\vec{F}$ taken over S is equal to the integral of divergence of $\vec{F}$ taken over V.

ie., $\iint_S \vec{F}.\hat{n}\,ds = \iiint_V \nabla.\vec{F}\,dv$ **Where $\hat{n}$ is the unit vector in the positive normal to S.**

21. Evaluate $\iint_S \vec{r}.\hat{n}\,ds$, where S is a Closed surface .

By Gauss Divergence theorem , we have

$$\iint_S \vec{r}.\hat{n}\,ds = \iiint_V \nabla.\vec{r}\,dv$$

$$= \iiint_V \left[\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z} \right]\left(x\,\vec{i} + y\,\vec{j} + z\,\vec{k} \right) dv$$

$$= \iiint_V \left[\frac{\partial(x)}{\partial x} + \frac{\partial(y)}{\partial y} + \frac{\partial(z)}{\partial z} \right] dv$$

$$= \iiint_V (1+1+1)dv = 3\iiint_V dv = 3V$$

22.. Prove that $\iint_S \phi.\hat{n}\,ds = \iiint_V \nabla.\vec{\phi}\,dV$

By Gauss Divergence theorem , we have $\iint_S \vec{F}.\hat{n}\,ds = \iiint_V \nabla.\vec{F}\,dV$

Let $\vec{F} = \phi\,\vec{c}$ **where** $\vec{c}$ **is a constant vector. Then ,**

$$\iint_S \vec{\phi}\,\vec{c}\,.\hat{n}\,ds = \iiint_V \nabla.(\phi\,\vec{c})dv$$

$$\iint_S \vec{c}.(\vec{\phi}\,\hat{n})ds = \iiint_V \vec{c}\,.(\nabla\phi)dv$$

Taking $\vec{c}$ **outside the integrals , we get**

$$\vec{c}\,.\iint_S \vec{\phi}.\hat{n}\,ds = \vec{c}\iiint_V \nabla\phi dv$$

$$\iint_S \vec{\phi}\,\hat{n}\,ds = \iiint_V \nabla\phi dv$$

23. Evaluate $\iint_s xdydz + ydzdx + zdxdy$ **over the region of radius a.**

$$\iint_s xdydz + ydzdx + zdxdy = \iiint_V \left[\frac{\partial(x)}{\partial x} + \frac{\partial(y)}{\partial y} + \frac{\partial(z)}{\partial z} \right] dxdydz$$

$$= \iiint_V (1+1+1)dxdydz$$

$$= 3\iiint_V dv = 3v$$

$$= 3\left[\frac{4}{3}\pi a^3\right] = 4\pi a^3$$

24. State Green's theorem in the plane

If R is a closed region of the xy-plane bounded by a simple closed curve C and if M and N are continuous derivatives in R, then $\int Mdx + Ndy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)dxdy$ **where C is travelled in the anti-clockwise direction.**

25. Using Green's theorem, prove that the area enclosed by a simple closed curve C is $\frac{1}{2}\int (xdy - ydx)dxdy$ **.**

Consider By Green's theorem,

$$\int Mdx + Ndy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)dxdy \text{ ....(1)}$$

Consider $\frac{1}{2}\int (xdy - ydx)dxdy = \int \frac{x}{2}dy - \frac{y}{2}dx = \int -\frac{y}{2}dx + \frac{x}{2}dy$

[since, $M = -\frac{y}{2};; N = \frac{x}{2}$ **]**

From (1), $\int -\frac{y}{2}dx + \frac{x}{2}dy = \iint_R \left[\frac{1}{2} - \left(-\frac{1}{2}\right)\right]dxdy$

$= \iint_R dxdy$ **=Area bounded by a closed curve 'C'**

26. State Stoke's theorem.

If $\vec{F}$ **is any continuous differentiable vector function and S is a surface enclosed by a curve C then,** $\int_C \vec{F}.d\vec{r} = \iint_S curl\,\vec{F}.\hat{n}ds$ **where** $\hat{n}$ **is the unit normal vector at any point of S.**

27.Using Stoke's theorem, prove that $\int_c \vec{r}.d\vec{r} = 0$ **.**

Given, $\int_c \vec{r}.d\vec{r}$ **where** $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$

$\therefore \int_c \vec{r}.d\vec{r} = \iint_s curl\vec{r}.\hat{n}$ **ds** **[$\because$ by Stoke's theorem]**

$$= 0 \quad \left[\because curl\vec{r} = \nabla x\vec{r} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y & z \end{vmatrix} = 0 \right]$$

28. Find the constants a,b,c so that, $\vec{F} = (x + 2y + az)\vec{i} + (bx - 3y - z)\vec{j} + (4x + cy + 2z)\vec{k}$ **is irrotational.**

Given $\nabla x\vec{F} = 0$

$$\text{ie.,} \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x + 2y + az & bx - 3y - z & 4x + cy + 2z \end{vmatrix} = 0$$

$$\Rightarrow \vec{i}\left[\frac{\partial}{\partial y}(4x + cy + 2z) - \frac{\partial}{\partial z}(bx - 3y - z)\right]$$
$$-\vec{j}\left[\frac{\partial}{\partial x}(4x + cy + 2z) - \frac{\partial}{\partial z}(x + 2y + az)\right]$$
$$+\vec{k}\left[\frac{\partial}{\partial x}(bx - 3y - z) - \frac{\partial}{\partial y}(x + 2y + az)\right] = 0$$

$$\Rightarrow \vec{i}[c + 1] - \vec{j}[4 - a] + \vec{k}[b - 2] = 0\vec{i} + 0\vec{j} + 0\vec{k}$$

$\Rightarrow$ **c + 1 = 0 4 - a = 0 b - 2 = 0**

$\Rightarrow$ **c = -1 ; a = 4 ; b = 2**

29. If $\vec{F} = x^2\vec{i} + xy^2\vec{j}$**, evaluate the line integral** $\int_c \vec{F}.d\vec{r}$ **from (0,0) to (1,1) along the path y = x.**

Given $\vec{F} = x^2\vec{i} + xy^2\vec{j}$ **, x = y**

dx = dy

$$\vec{r} = x\vec{i} + y\vec{j}$$

$$d\vec{r} = dx\vec{i} + dy\vec{j}$$

$$\vec{F}.d\vec{r} = x^2dx + xy^2dy = x^2dx + x^3dx \qquad [\because x = y, dx = dy]$$

$$= (x^2 + x^3)dx$$

$$\int_c \vec{F}.d\vec{r} = \int_0^1 (x^2 + x^3)dx = \frac{7}{12}$$

30. What is the greatest rate of increase of $\phi = xyz^2$ at (1,0,3).

Given $\phi = xyz^2$

$$\nabla\phi = \vec{i}\frac{\partial\phi}{\partial x} + \vec{j}\frac{\partial\phi}{\partial y} + \vec{k}\frac{\partial\phi}{\partial z}$$

$$= \vec{i}(yz^2) + \vec{j}(xz^2) + \vec{k}(2xyz)$$

$$(\nabla\phi)_{(1,0,3)} = \vec{i}(yz^2) + \vec{j}(xz^2) + \vec{k}(2xyz)$$

The greatest rate of increase = $|\nabla\phi| = \sqrt{81} = 9$ **units**

31. Using Green's theorem , find the area of a circle of radius r.

We know by Green's theorem,

Area = $\frac{1}{2}\int_c (xdy - ydx)$

For a circle of radius r, we have $x^2 + y^2 = r^2$

Put $x = r\cos\theta, y = r\sin\theta$

$dx = -r\cos\theta d\theta, dy = r\sin\theta d\theta$ **[θ varies from 0 to 2π]**

Area $= \frac{1}{2}\int_0^{2\pi} [r\cos\theta r\cos\theta - r\sin\theta(-r\sin\theta)]d\theta$

$$= \frac{1}{2}\int_0^{2\pi} r^2 d\theta = \; = \frac{1}{2}r^2[\theta]_0^{2\pi}$$

Area = πr^2 **sq.units.**

32. If $\nabla\phi$ is solenoidal find $\nabla^2\phi$.

Given $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$ **is solenoidal.**

$\therefore \nabla.\nabla\phi = 0$

But $\nabla^2\phi = \nabla.\nabla\phi = 0$

33. If $\vec{r} = \left(x\vec{i} + y\vec{j} + z\vec{k} \right)$**, find** $\nabla \times \vec{r}$

Given $\vec{r} = \left(x\vec{i} + y\vec{j} + z\vec{k} \right)$

$$\nabla \times \vec{r} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial zx} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y & z \end{vmatrix} = \vec{i}(0-0) + \vec{j}(0-0) + \vec{k}(0-0) = \vec{0}$$

34. Define Volume integral.

An integral which can be evaluated over a volume closed by a surface is called a volume integral. Volume integral can be evaluated by triple integration. Ie., $\iiint_v f(x, y, z)dv$

35. State Gauss Divergence theorem.

If $\vec{F}$ **is a vector point function, finite and differentiable in a region r bounded by a closed surface S, then the surface integral of the normal component of** $\vec{F}$ **taken over S is equal to the integral of divergence of** $\vec{F}$ **taken over V.**

ie., $\iint_S \vec{F}.\hat{n}\,ds = \iiint_V \nabla.\vec{F}\,dv$ **Where** $\hat{n}$ **is the unit vector in the positive normal to S.**

36.Evaluate $\iint_S \vec{r}.\hat{n}\,ds$**, where S is a Closed surface .**

By Gauss Divergence theorem , we have

$$\iint_S \vec{r}.\hat{n}\,ds = \iiint_V \nabla.\vec{r}\,dv$$

$$= \iiint_V \left[\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z} \right] \left(x\,\vec{i} + y\,\vec{j} + z\,\vec{k} \right) dv$$

$$= \iiint_V \left[\frac{\partial (x)}{\partial x} + \frac{\partial (y)}{\partial y} + \frac{\partial (z)}{\partial z} \right] dv$$

$$= \iiint_V (1+1+1)dv = 3\iiint_V dv = 3V$$

37. Prove that $\iint_S \phi.\hat{n}\,ds = \iiint_V \nabla.\vec{\phi}\,dV$ **By Gauss Divergence theorem , we have**

$$\iint_S \vec{F}.\hat{n}\,ds = \iiint_V \nabla.\vec{F}\,dV$$

Let $\vec{F} = \phi\,\vec{c}$ **where** $\vec{c}$ **is a constant vector. Then ,**

$$\iint_S \vec{\phi}\,\vec{c}\,.\hat{n}\,ds = \iiint_V \nabla.(\phi\,\vec{c})dv$$

$$\iint_S \vec{c}.(\vec{\phi}\,\hat{n})ds = \iiint_V \vec{c}\,.(\nabla\phi)dv$$

Taking $\vec{c}$ **outside the integrals , we get**

$$\vec{c}\,.\iint_S \vec{\phi}.\hat{n}\,ds = \vec{c}\iiint_V \nabla\phi dv$$

$$\iint_S \vec{\phi}\,\hat{n}\,ds = \iiint_V \nabla\phi dv$$

38. Evaluate $\iint_s xdydz + ydzdx + zdxdy$ **over the region of radius a.**

$$\iint_s xdydz + ydzdx + zdxdy = \iiint_V \left[\frac{\partial (x)}{\partial x} + \frac{\partial (y)}{\partial y} + \frac{\partial (z)}{\partial z} \right] dxdydz$$

$$= \iiint_V (1 + 1 + 1) dxdydz$$

$$= 3\iiint_V dv = 3v$$

$$= 3\left[\frac{4}{3}\pi a^3\right] = 4\pi a^3$$

39. State Green's theorem in the plane

If R is a closed region of the xy-plane bounded by a simple closed curve C and if M and N are continuous derivatives in R, then

$\int Mdx + Ndy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dxdy$ **where C is travelled in the anti-clockwise direction.**

40. Using Green's theorem , prove that the area enclosed by a simple closed curve C

is $\frac{1}{2}\int (xdy - ydx) dxdy$ **.**

consider By Green's theorem,

$$\int Mdx + Ndy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dxdy \;\ldots\ldots\ldots\ldots (1)$$

Consider $\frac{1}{2}\int (xdy - ydx) dxdy = \int \frac{x}{2}dy - \frac{y}{2}dx = \int -\frac{y}{2}dx + \frac{x}{2}dy$

[since, $M = -\frac{y}{2};; N = \frac{x}{2}$ **]**

From (1), $\int -\frac{y}{2}dx + \frac{x}{2}dy = \iint_R \left[\frac{1}{2} - \left(-\frac{1}{2}\right)\right] dxdy$

$= \iint_R dxdy$ **=Area bounded by a closed curve 'C'**

41. State Stoke's theorem.

If $\vec{F}$ is any continuous differentiable vector function and S is a surface enclosed by a curve C then, $\int_C \vec{F}.d\vec{r} = \iint_S curl\,\vec{F}.\hat{n}ds$ where $\hat{n}$ is the unit normal vector at any point of S.

42. If $\vec{F} = (y^2\cos x + z^2)\vec{i} + (2y\sin x - 4)\vec{j} + 3xz^2\vec{k}$,find its scalar potential.

To find ϕ such that $\vec{F} = grad\phi$

$$(y^2\cos x + z^2)\vec{i} + (2y\sin x - 4)\vec{j} + 3xz^2\vec{k} = \vec{i}\frac{\partial\phi}{\partial x} + \vec{j}\frac{\partial\phi}{\partial y} + \vec{k}\frac{\partial\phi}{\partial z}$$

Integrating the equations partially w.r.to x,y,z respectively.

$$\phi = y^2\sin x + xz^3 + f_1(y,z)$$
$$\phi = y^2\sin x - 4y + f_2(x,z)$$
$$\phi =^2 xz^3 + f_3(y,z$$

Therefore $\phi = y^2\sin x + xz^3 - 4y + c$ is a scalar potential.