Unit - 3

Introduction

If x & y are real numbers then $z = x+iy$ is called a complex number where x is called real part of z, y is called imaginary part of z and the value of i is $\sqrt{-1}$. The complex number $x-iy$ is called the complex conjugate of z and it is denoted by $\bar{z}$. ie) $\bar{z} = x-iy$

Note :

1. $|z| = \sqrt{x^2+y^2}$
2. $|z^2| = z\bar{z}$
3. $z\bar{z} = x^2+y^2 = r^2$
4. $|\bar{z}| = |z|$
5. Real part of $z = \frac{z+\bar{z}}{2}$
6. Imaginary part of $z = \frac{z-\bar{z}}{2}$
7. $z = re^{i\theta}$ is called polar form of z

Function of complex variable

$w = f(z) = u(x,y) + iv(x,y)$ where $u(x,y)$ & $v(x,y)$ are real variables.

Analytic function

A function is said to be analytic at a point if its derivative exists not only that point but also some neighbourhood of that point

1. Show that the function $f(z) = \bar{z}$ is nowhere differentiable.

Soln.

Given $f(z) = \bar{z} = x - iy$

$u + iv = x - iy$

$\Rightarrow u = x$ and $v = -y$

$u_x = 1$ $\quad$ $v_x = 0$

$u_y = 0$ $\quad$ $v_y = -1$

Here $u_x \neq v_y$ and $u_y = -v_x$

Hence C-R eqns are not satisfied.

$\Rightarrow f(z) = \bar{z}$ is not differentiable anywhere (or) nowhere differentiable.

2. Determine whether the function $2xy + i(x^2 - y^2)$ is analytic or not.

Soln.

Let $f(z) = 2xy + i(x^2 - y^2)$

$u + iv = 2xy + i(x^2 - y^2)$

$\Rightarrow u = 2xy$ and $v = x^2 - y^2$

$u_x = 2y$ $\quad$ $v_x = 2x$

$u_y = 2x$ $\quad$ $v_y = -2y$

$\Rightarrow u_x \neq v_y$ and $u_y \neq -v_x$

C.R eqns. are not satisfied.

Hence $f(z)$ is not an analytic function.

3. Let $f(z) = z^3$ be analytic. Justify.

Soln.

$$u+iv = [x^3 - 3xy^2] + i[3x^2y - y^3]$$

$\Rightarrow u = x^3 - 3xy^2$ and $v = 3x^2y - y^3$

$u_x = 3x^2 - 3y^2$ $\qquad v_x = 6xy$

$u_y = -6xy$ $\qquad v_y = -3y^2 + 3x^2$

$\Rightarrow u_x = v_y$ and $u_y = -v_x$

CR eqns are satisfied.

Hence, $f(z)$ is analytic.

4. Find the constants a, b, c if $f(z) = x + ay + i(bx + cy)$ is analytic.

Soln.

Let $f(z) = x + ay + i(bx + cy)$

$u + iv = x + ay + i(bx + cy)$

Here $u = x + ay$ and $v = bx + cy$

$u_x = 1$ $\qquad v_x = b$

$u_y = a$ $\qquad v_y = c$

Since $f(z)$ is analytic.

$\Rightarrow u_x = v_y$ and $u_y = -v_x$

$1 = c$ $\qquad a = -b$

$\therefore a = -b$ and $c = 1$.