



Necessary condition for $f(z)$ to be analytic:

If $w = f(z) = u + iv$ is an analytic function, then Cauchy-Riemann eqns are satisfied.

$$\text{i.e., } \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$$

$$\Rightarrow u_x = v_y \quad \text{and} \quad v_x = -u_y$$

Sufficient condition for Analytic function:

If the partial derivatives u_x, u_y, v_x and v_y are all continuous and $u_x = v_y$ and $u_y = -v_x$, then the function is analytic.

1. Show that the function $f(z) = \bar{z}$ is nowhere differentiable.

Soln.

$$\text{Given } f(z) = \bar{z} = x - iy$$

$$u + iv = x - iy$$

$$\Rightarrow u = x \quad \text{and} \quad v = -y$$

$$u_x = 1 \quad v_x = 0$$

$$u_y = 0 \quad v_y = -1$$

Here $u_x \neq v_y$ and $u_y \neq -v_x$

Hence C-R eqns are not satisfied.

$\Rightarrow f(z) = \bar{z}$ is not differentiable anywhere (or) nowhere differentiable.



2. Determine whether the function $2xy + i(x^2 - y^2)$ is analytic or not.

Soln.

$$\text{Let } f(z) = 2xy + i(x^2 - y^2)$$

$$u + iv = 2xy + i(x^2 - y^2)$$

$$\Rightarrow u = 2xy \quad \text{and} \quad v = x^2 - y^2$$

$$u_x = 2y$$

$$v_x = 2x$$

$$u_y = 2x$$

$$v_y = -2y$$

$$\Rightarrow u_x \neq v_y \quad \text{and} \quad u_y \neq -v_x$$

C.R eqns. are not satisfied.

Hence $f(z)$ is not an analytic function.

3. Let $f(z) = z^3$ be analytic. Justify.

Soln.

$$\text{Let } f(z) = z^3$$

$$u + iv = (x + iy)^3$$

$$= x^3 + 3x^2(iy) + 3x(iy)^2 + (iy)^3$$

$$= x^3 + i3x^2y - 3xy^2 - iy^3$$



$$u+iv = [x^3 - 3xy^2] + i [3x^2y - y^3]$$

$$\Rightarrow u = x^3 - 3xy^2 \quad \text{and} \quad v = 3x^2y - y^3$$

$$u_x = 3x^2 - 3y^2 \quad v_x = 6xy$$

$$u_y = -6xy \quad v_y = -3y^2 + 3x^2$$

$$\Rightarrow u_x = v_y \quad \text{and} \quad u_y = -v_x$$

CR eqns are satisfied.
Hence $f(z)$ is analytic.

4. Find the constants a, b, c if $f(z) = x+ay+i(bx+cy)$ is analytic.

Soln.

$$\text{Let } f(z) = x+ay+i(bx+cy)$$

$$u+iv = x+ay+i(bx+cy)$$

Here $u = x+ay$ and $v = bx+cy$

$$u_x = 1 \quad v_x = b$$

$$u_y = a \quad v_y = c$$

Since $f(z)$ is analytic.

$$\Rightarrow u_x = v_y \quad \text{and} \quad u_y = -v_x$$

$$1 = c \quad a = -b$$

$$\therefore a = -b \quad \text{and} \quad c = 1.$$

5. Check whether the function $w = \sin z$ is analytic or not.

Soln.

$$\text{Let } w = f(z) = \sin z$$

$$u+iv = \sin(x+iy)$$

$$= \sin x \cos iy + \cos x \sin iy$$

$$u+iv = \sin x \cosh y + i \cos x \sinh y$$

Here $u = \sin x \cosh y$ and $v = \cos x \sinh y$

$$u_x = \cos x \cosh y \quad v_x = -\sin x \sinh y$$

$$u_y = \sin x \sinh y \quad v_y = \cos x \cosh y$$

$$\Rightarrow u_x = v_y \quad \text{and} \quad u_y = -v_x$$

$$\begin{cases} \cos iy = \cosh y \\ \sin iy = i \sinh y \end{cases}$$



CR eqns are satisfied.
 Also the 4 partial derivatives are continuous.
 Hence the function is analytic.

Properties of Analytic function:

i) $\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0$ is known as the Laplace eqn in two dimensions.

Property 1:

The real and imaginary part of an analytic fn. $w = u + iv$ satisfies Laplace eqn.

Proof:

Let $w = f(z) = u + iv$ be analytic.

To prove u and v satisfies Laplace eqn.

$$\text{i.e., } \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad \text{and} \quad \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0$$

Since $f(z)$ is analytic.

$$\Rightarrow u_x = v_y \quad \text{and} \quad u_y = -v_x \rightarrow (1)$$

Differentiate (1) partially w.r. to x and y ,

$$\frac{\partial}{\partial x} \left(\frac{\partial u}{\partial x} \right) = \frac{\partial}{\partial x} \left(\frac{\partial v}{\partial y} \right)$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 v}{\partial x \partial y} \rightarrow (2)$$

$$\text{and } \frac{\partial^2 u}{\partial y^2} = -\frac{\partial^2 v}{\partial y \partial x} \rightarrow (3)$$

$$(2) + (3) \Rightarrow \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 v}{\partial x \partial y} - \frac{\partial^2 v}{\partial y \partial x}$$

$= 0$
 $\Rightarrow u$ satisfies Laplace eqn.

Differentiate (1) partially w.r. to y and x

$$\frac{\partial}{\partial y} \left(\frac{\partial u}{\partial x} \right) = \frac{\partial}{\partial y} \left(\frac{\partial v}{\partial y} \right) \quad \text{and} \quad \frac{\partial}{\partial x} \left(\frac{\partial u}{\partial y} \right) = -\frac{\partial}{\partial x} \left(\frac{\partial v}{\partial x} \right)$$



$$\begin{aligned} \frac{\partial^2 u}{\partial y \partial x} &= \frac{\partial^2 v}{\partial y^2} & \left\{ \begin{aligned} \frac{\partial^2 u}{\partial x \partial y} &= -\frac{\partial^2 v}{\partial x^2} \\ \frac{\partial^2 v}{\partial x^2} &= -\frac{\partial^2 u}{\partial x \partial y} \end{aligned} \right. \\ \frac{\partial^2 v}{\partial y^2} &= \frac{\partial^2 u}{\partial y \partial x} & \begin{aligned} \hookrightarrow (4) \\ \hookrightarrow (5) \end{aligned} \end{aligned}$$

$$(4) + (5) \Rightarrow \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = \frac{\partial^2 u}{\partial y \partial x} - \frac{\partial^2 u}{\partial x \partial y}$$

$$= 0$$

$\Rightarrow v$ satisfies Laplace eqn.
Hence u & v satisfies Laplace eqn.

Property 2:

Show that an analytic function with a real part is constant.

Proof:

Let $f(z)$ be analytic function.

$$\Rightarrow u_x = v_y \text{ and } u_y = -v_x \rightarrow (1)$$

Given $u = \text{constant} = c$ (say)

Differentiate partially w.r. to x & y ,

$$\frac{\partial u}{\partial x} = 0 \text{ and } \frac{\partial u}{\partial y} = 0$$

$$\therefore, u_x = 0 \quad u_y = 0$$

$$\therefore, \text{from (1)} \quad v_y = 0 \quad -v_x = 0 \Rightarrow v_x = 0$$

$$\therefore f'(z) = \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} = 0 + i(0)$$

$$= 0 + i(0)$$

$$f'(z) = 0$$

$$\Rightarrow f(z) = c$$

$\therefore f(z)$ is constant.

Property 3:

An analytic function with constant modulus is analytic.