



SNS COLLEGE OF TECHNOLOGY

(An Autonomous Institution)

Coimbatore-641035.



UNIT-II COMPLEX DIFFERENTIATION

Construction of Analytic functions

Construction of Analytic functions:

Method of Thomson method

i). To find $f(z)$, when u is given

$$f(z) = \int [\phi_1(z, 0) - i\phi_2(z, 0)] dz$$

$$\text{where } \phi_1(z, 0) = \left(\frac{\partial u}{\partial x} \right)_{(z, 0)} \text{ and}$$

$$\phi_2(z, 0) = \left(\frac{\partial u}{\partial y} \right)_{(z, 0)}$$

ii). To find $f(z)$, when v is given

$$f(z) = \int [\phi_1(z, 0) + i\phi_2(z, 0)] dz$$

$$\text{where } \phi_1(z, 0) = \left(\frac{\partial v}{\partial y} \right)_{(z, 0)} \text{ and}$$

$$\phi_2(z, 0) = \left(\frac{\partial v}{\partial x} \right)_{(z, 0)}$$

iii). If $u-v$ or $u+v$ is given, then to find $f(z)$

$$\text{Take } f(z) = u + iv$$

$$\text{if } f(z) = iu - v$$

i). Find the analytic function $f(z)$ whose real part is $u = 3x^2y + 2x^2 - y^3 - 2y^2$

Soln.

$$\text{Given } u = 3x^2y + 2x^2 - y^3 - 2y^2$$

$$\frac{\partial u}{\partial x} = 6xy + 4x$$



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$$\phi_1(x, 0) = \left(\frac{\partial u}{\partial x} \right)_{(x, 0)} = 4x$$

$$\frac{\partial u}{\partial y} = 3x^2 - 3y^2 - 1y$$

$$\phi_2(x, 0) = \left(\frac{\partial u}{\partial y} \right)_{(x, 0)} = 3x^2$$

By Milne Thompson method,

$$f(z) = \int [\phi_1(z, 0) - i\phi_2(z, 0)] dz$$

$$= \int [4x - i3x^2] dz$$

$$= \frac{4z^2}{2} - i \frac{3z^3}{3} + C$$

$$f(z) = 2z^2 - iz^3 + C$$

$$\text{Hence } 2x^2 - 3xy^2 + 3x^3 - iy^3 + C$$

$$\text{Ans: } z^2 - 3z^2 + C$$

Q. Prove that $v = e^{-x}(\alpha \cos y + y \sin y)$ is harmonic and determine analytic function $f(z)$.

Soln.

$$\text{Given } v = e^{-x} \alpha \cos y + e^{-x} y \sin y$$

$$\frac{\partial v}{\partial x} = [e^{-x} + x(-e^{-x})J \cos y + (-1)e^{-x} y \sin y]$$

$$= e^{-x} \cos y - x e^{-x} \cos y - e^{-x} y \sin y$$

$$\frac{\partial^2 v}{\partial x^2} = -e^{-x} \cos y - [x(-e^{-x}) + e^{-x}] J \cos y + e^{-x} y \sin y$$

$$= -e^{-x} \cos y + x e^{-x} \cos y - e^{-x} \cos y + e^{-x} y \sin y$$

$$\frac{\partial v}{\partial y} = e^{-x} x [-\sin y] + e^{-x} [y \cos y + \sin y]$$

$$= -x e^{-x} \sin y + e^{-x} y \cos y + e^{-x} \sin y$$

$$\frac{\partial^2 v}{\partial y^2} = -x e^{-x} \cos y + e^{-x} [y (-\sin y) + \cos y]$$

$$+ e^{-x} \cos y$$

$$= -x e^{-x} \cos y - y \sin y e^{-x} + e^{-x} \cos y$$

$$+ e^{-x} \cos y$$



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$$\begin{aligned}
 &= -x e^{-x} \cos y - e^{-x} y \sin y + e^{-x} \cos y \\
 \therefore \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} &= -e^{-x} \cos y + x e^{-x} \cos y + e^{-x} y \sin y \\
 &\quad - x e^{-x} \cos y - e^{-x} y \sin y + e^{-x} \cos y \\
 &= 0
 \end{aligned}$$

Hence v satisfies Laplace eqn.

v is harmonic.

By Milne's Thomson method,

$$f(z) = \int [\phi_1(z, 0) + i \phi_2(z, 0)] dz$$

$$\text{where } \phi_1(z, 0) = \left(\frac{\partial v}{\partial y} \right)_{(z, 0)}$$

$$= [-x e^{-x} \sin y + e^{-x} y \cos y + e^{-x} \sin y]_{(z, 0)}$$

$$\phi_1(z, 0) = 0$$

$$\phi_2(z, 0) = \left(\frac{\partial v}{\partial x} \right)_{(z, 0)}$$

$$= [e^{-x} \cos y - x e^{-x} \cos y - e^{-x} y \sin y]_{(z, 0)}$$

$$\phi_2(z, 0) = e^{-x} - x e^{-x}$$

$$\therefore f(z) = \int [0 + i(e^{-x} - x e^{-x})] dz$$

$$= i \left[\int e^{-x} dx - \int x e^{-x} dx \right]$$

$$= i \left[-e^{-x} - (-x e^{-x} - e^{-x}) \right] + c$$

$$= i \left[-e^{-x} + x e^{-x} + e^{-x} \right] + c$$

$$\underline{f(z) = i x e^{-x} + c}$$

$$\begin{aligned}
 \text{Now } v &= e^{2x} (\cos 2y + i \sin 2y) \\
 f(z) &= ?
 \end{aligned}$$

$$v = x^2 - y^2 + 2i$$



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UNIT-II COMPLEX DIFFERENTIATION

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By Milne's Thompson method,

$$\begin{aligned} F(z) &= \int [\phi_1(z, 0) - i\phi_2(z, 0)] dz \\ &= \int [e^z + ie^z] dz \\ &= (1+i) \int e^z dz \end{aligned}$$

$$(1+i) f(z) = (1+i) e^z + C$$

$$f(z) = e^z + C$$

5]. If $f(z) = u+iv$ is analytic, find $f(z)$
given that $u+v = \frac{\sin 2x}{\cos 2y - \cos 2x}$

Soln.

$$\text{Let } f(z) = u+iv \rightarrow (1)$$

$$i f(z) = iv - v \rightarrow (2)$$

$$(1) + (2) \Rightarrow (1+i) f(z) = u+iv + iv - v$$

$$= (u-v) + i(u+v)$$

$$f(z) = u+iv$$

$$\text{Hence } F(z) = (1+i) f(z).$$

$$v = u - v$$

$$\therefore v = u + v$$

$$\text{Given } v = u + v = \frac{\sin 2x}{\cos 2y - \cos 2x}$$

$$\frac{\partial v}{\partial x} = \frac{(\cos 2y - \cos 2x)(2\cos 2x) - \sin 2x}{(\cos 2y - \cos 2x)^2}$$

$$\begin{aligned} \phi_2(z, 0) &= \left(\frac{\partial v}{\partial x} \right)_{(z, 0)} = \frac{(1 - \cos 2x) 2 \cos 2x - 2 \sin^2 2x}{(1 - \cos 2x)^2} \\ &= \frac{(1 - \cos 2x)^2 \cos 2x - 2(1 - \cos^2 2x)}{(1 - \cos 2x)^2} \\ &= \frac{(1 - \cos 2x)^2 \cos 2x - 2(1 + \cos 2x)(1 - \cos 2x)}{(1 - \cos 2x)^2} \end{aligned}$$



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By Milne's Thomson method,

$$\begin{aligned} f(z) &= \int [\phi_1(z, 0) - i\phi_2(z, 0)] dz \\ &= \int [-\csc^2 z - i(0)] dz \\ &= - \int \csc^2 z dz \\ f(z) &= \cot z + C \end{aligned}$$

Q. Find the analytic function $f(z) = u + iv$
where $u - v = e^x(\cos y - \sin y)$
Soln.

$$\begin{aligned} \text{Let } f(z) &= u + iv \rightarrow (1) \\ i f(z) &= iu - v \rightarrow (2) \\ (1) + (2) \Rightarrow (1+i) f(z) &= u + iv + iu - v \\ (1+i) f(z) &= (u - v) + i(u + v) \\ F(z) &= u + iv \end{aligned}$$

$$\text{Here } F(z) = (1+i) f(z)$$

$$u = u - v$$

$$v = u + v$$

$$\text{Given } u = u - v = e^x (\cos y - \sin y)$$

$$\frac{\partial u}{\partial x} = e^x [\cos y - \sin y]$$

$$\phi_1(z, 0) = \left(\frac{\partial u}{\partial x} \right)_{(z, 0)} = e^x [1 - 0] = e^x$$

$$\begin{aligned} \left(\frac{\partial u}{\partial y} \right)_{(z, 0)} &= e^x [-\sin y - \cos y] \\ &= -e^x [\sin y + \cos y] \end{aligned}$$

$$\phi_2(z, 0) = \left(\frac{\partial v}{\partial y} \right)_{(z, 0)} = -e^x [0 + 1] = -e^x$$



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UNIT-II COMPLEX DIFFERENTIATION

Construction of Analytic functions

3. + Determine the analytic function whose real part is $\frac{\sin 2x}{\cosh 2y - \cos 2x}$

Soln.

$$\text{Given } u = \frac{\sin 2x}{\cosh 2y - \cos 2x}$$

$$\frac{\partial u}{\partial x} = \frac{(\cosh 2y - \cos 2x) 2 \cos 2x - \sin 2x(2 \sin 2x)}{(\cosh 2y - \cos 2x)^2}$$

$$\begin{aligned}\phi_1(x,0) &= \left(\frac{\partial u}{\partial x}\right)_{(x,0)} = \frac{(1 - \cos 2x) 2 \cos 2x - 2 \sin^2 2x}{(1 - \cos 2x)^2} \\ &= \frac{(1 - \cos 2x) 2 \cos 2x - 2(1 - \cos^2 2x)}{(1 - \cos 2x)^2} \\ &= \frac{(1 - \cos 2x) 2 \cos 2x - 2(1 + \cos 2x)(1 - \cos 2x)}{(1 - \cos 2x)^2} \\ &= \frac{2[\cos 2x - (1 + \cos 2x)]}{(1 - \cos 2x)} \\ &= \frac{-2}{(1 - \cos 2x)} \\ &= \frac{-1}{1 - \cos 2x} = \frac{-1}{\sin^2 x}\end{aligned}$$

$$\phi_1(x,0) = -\csc^2 x$$

$$\begin{aligned}\text{and } \frac{\partial u}{\partial y} &= \sin 2x \frac{\partial}{\partial y} \left[\frac{1}{\cosh 2y - \cos 2x} \right] \\ &= \sin 2x \frac{-1}{[\cosh 2y - \cos 2x]^2} [\cancel{2 \sinh 2y} - 0] \\ &= -\frac{2 \sin 2x \sinh 2y}{(\cosh 2y - \cos 2x)^2} \\ \phi_2(x,0) &= \left(\frac{\partial u}{\partial y}\right)_{(x,0)} = \frac{-2 \sin 2x(0)}{(1 - \cos 2x)^2} = 0\end{aligned}$$



$$\begin{aligned}
 &= \frac{\alpha \cos \alpha z - \alpha (1 + \cos \alpha z)}{1 - \cos \alpha z} \\
 &= \frac{\alpha [\cos \alpha z - 1 - \cos \alpha z]}{1 - \cos \alpha z} \\
 &= \frac{-\alpha}{1 - \cos \alpha z} = \frac{-1}{\frac{1 - \cos \alpha z}{\alpha}} = \frac{-1}{\sin^2 z}
 \end{aligned}$$

$$\phi_2(z, 0) = -\csc^2 z$$

$$\begin{aligned}
 \frac{\partial v}{\partial y} &= \sin \alpha x \frac{-1}{(\cosh \alpha y - \cos \alpha x)^2} [2 \sinh \alpha y - 0] \\
 &= -\frac{2 \sin \alpha x \sinh \alpha y}{(\cosh \alpha y - \cos \alpha x)^2}
 \end{aligned}$$

$$\phi_1(z, 0) = \left(\frac{\partial v}{\partial y} \right)_{(z, 0)} = 0$$

By milne's Thomson method,

$$\begin{aligned}
 F(z) &= \int [\phi_1(z, 0) + i \phi_2(z, 0)] dz \\
 &= \int [0 + i(-\csc^2 z)] dz \\
 &= -i \int \csc^2 z dz
 \end{aligned}$$

$$F(z) = i \cot z + C$$

$$(1+i) f(z) = i \cot z + C$$

$$f(z) = \frac{i}{1+i} \cot z + \frac{C}{1+i}$$

$$= \frac{i(1-i)}{(1+i)(1-i)} \cot z + C$$

$$= \frac{i - i^2}{1 - i^2} \cot z + C$$

$$f(z) = \frac{(1+i)}{2} \cot z + C$$