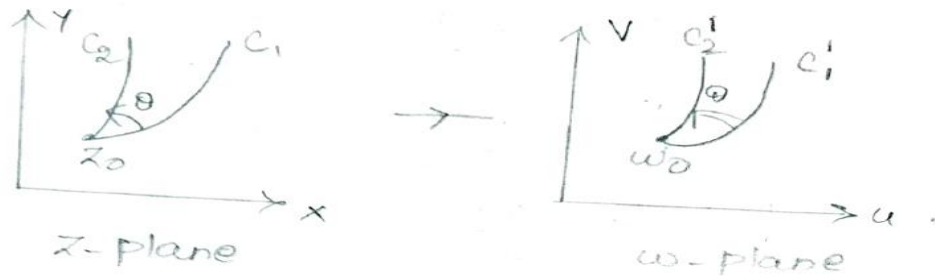## Conformal mapping:

A transformation that preserves angles between every pair of curves through a point both in magnitude and sense, is said to be conformal at that point.

z-plane   w-plane

Isogonal mapping:

A transformation under which angles between every pair of curves through a point are preserved in magnitude, but different in direction.

z-plane   w-plane

Critical point:

The point z_0 is said to be critical point of the transformation $w = f(z)$ if $f(z)$ is not conformal at z_0. The critical point is defined by,

$$\frac{dw}{dz} = 0 \quad and \quad \frac{dz}{dw} = 0$$

1]. Find the critical point of the transformation $\omega = z + \dfrac{1}{z}$

Soln.

Given $\omega = z + \dfrac{1}{z}$

$$\frac{d\omega}{dz} = 1 - \frac{1}{z^2} = \frac{z^2 - 1}{z^2}$$

$$\frac{dz}{d\omega} = \frac{z^2}{z^2 - 1}$$

Now $\dfrac{d\omega}{dz} = 0 \Rightarrow \dfrac{z^2 - 1}{z^2} = 0$

$$z^2 - 1 = 0$$
$$z^2 = 1$$
$$z = \pm 1$$

and $\dfrac{dz}{d\omega} = 0 \Rightarrow \dfrac{z^2}{z^2 - 1} = 0 \Rightarrow z^2 = 0$

$$z = 0$$

The critical points are $0, \pm 1$.

2]. Find the critical point of $\omega^2 = (z - \alpha)(z - \beta)$

Soln.

Given $\omega^2 = (z - \alpha)(z - \beta)$

Differentiate w.r. to z,

$$2\omega \frac{d\omega}{dz} = (z - \alpha)(1) + (z - \beta)(1)$$
$$= (z - \alpha) + (z - \beta)$$

$$\frac{d\omega}{dz} = \frac{2z - (\alpha + \beta)}{2\omega}$$

and $\dfrac{dz}{d\omega} = \dfrac{2\omega}{2z - (\alpha + \beta)}$

Now $\dfrac{d\omega}{dz} = 0 \Rightarrow \dfrac{2z - (\alpha + \beta)}{2\omega} = 0$

$$2z = \alpha + \beta$$
$$z = \frac{\alpha + \beta}{2}$$

and $\quad \dfrac{dz}{dw} = 0 \Rightarrow \dfrac{2w}{2z - (\alpha+\beta)} = 0$

$$2w = 0$$
$$w = 0$$
$$\sqrt{(z-\alpha)(z-\beta)} = 0$$
$$(z-\alpha)(z-\beta) = 0$$
$$z = \alpha \quad \text{and} \quad z = \beta$$

$\therefore$ The critical points are $z = \dfrac{\alpha+\beta}{2}, \alpha, \beta$.

3] Find the critical point of $w = \overline{z^4 - 1}$

Soln.

Given $w = z^4 - 1$,

Now, $\dfrac{dw}{dx} = 4z^3 \qquad \Rightarrow \qquad \dfrac{dw}{dz} = 0 \Rightarrow 4z^3 = 0$
$$z = 0$$

and $\dfrac{dz}{dw} = \dfrac{1}{4z^3} = 0$, which is not possible

$\therefore$ The critical point is 0.

4] Find the critical point of $w = \overline{\sin z}$

Soln.

Given $w = \sin z$

$\dfrac{dw}{dz} = \cos z \qquad \Rightarrow \qquad \dfrac{dw}{dx} = 0$

$$\cos z = 0$$
$$z = \cos^{-1}(0)$$
$$z = \pm \dfrac{\pi}{2}, \pm \dfrac{3\pi}{2}, \pm \dfrac{5\pi}{2}, \dots$$
$$z = \pm (2n-1)\dfrac{\pi}{2} \quad \text{where} \quad n = 1,2,3,\dots$$

and $\dfrac{dz}{dw} = 0 \Rightarrow \dfrac{1}{\cos z} = 0$, which is not possible.

Fixed point:

A point $z = a$ is said to be fixed point of a mapping $w = f(x)$ if its image under $f(x)$ is itself.

i.e., $f(z) = z$

1] Find the invariant points of $w = \dfrac{1+z}{1-z}$

Soln.

Given $w = \dfrac{1+z}{1-z}$

The invariant points are,

$z = \dfrac{1+z}{1-z}$ $(\because w = f(z) = z)$

$z(1-z) = 1+z$

$z - z^2 = 1 + z$

$z - z^2 - 1 - z = 0$

$-z^2 - 1 = 0$

$z^2 = -1$

$z = \pm i$

2]. Find the fixed points of $w = \dfrac{2z+6}{z+7}$

Soln.

Given $w = \dfrac{2z+6}{z+7}$

The fixed points are,

$z = \dfrac{2z+6}{z+7}$

$z(z+7) = 2z+6$

$z^2 + 7z - 2z - 6 = 0$

$z^2 + 5z - 6 = 0$

$(z-1)(z+6) = 0$

$z = 1, -6$

Hw $w = \frac{1}{2}$