Principal Stresses and Strains

In the first unit, we have studied in detail, the direct tensile and compressive stress as well as simple shear. In these chapters, we have always referred the stress in a plane, which is at right angles to the line of action of the force (in case of direct tensile or compressive stress). Moreover, we have considered at a time one type of stress, acting in one direction only. But the majority of engineering, component and structures are subjected to such loading conditions (or sometimes are of such shapes) that there exists a complex state of stresses; involving direct tensile and compressive stress as well as shear stress in various directions.

Principal Planes

It has been observed that at any point in a strained material, there are three planes, mutually perpendicular to each other, which carry direct stresses only, and no shear stress. A little consideration will show that out of these three direct stresses one will be maximum, the other minimum, and the third an intermediate between the two. These particular planes, which have no shear stress, are known as *principal planes*.

Principal Stress

The magnitude of direct stress, across a principal plane, is known as principal stress. The determination of principal planes, and then principal stress is an important factor in the design of various structures and machine components.

Methods for the Stresses on an Oblique Section of a Body

The following two methods for the determination of stresses on an oblique section of a strained body are important from the subject point of view :

1. Analytical method and 2. Graphical method.

Analytical Method for the Stresses on an Oblique Section of a Body

Here we shall first discuss the analytical method for the determination of stresses on an oblique section in the following cases, which are important from the subject point of view :

1. A body subjected to a direct stress in one plane.
2. A body subjected to direct stresses in two mutually perpendicular directions.

Sign Conventions for Analytical Method

Though there are different sign conventions, used in different books, yet we shall adopt the following sign conventions, which are widely used and internationally recognised :

1. All the tensile stresses and strains are taken as positive, whereas all the compressive stresses and strains are taken as negative.
2. The well established principles of mechanics is used for the shear stress. The shear stress which tends to rotate the element in the clockwise direction is taken as positive, whereas that which tends to rotate in an anticlockwise direction as negative.

In the element shown in Fig. 7.1, the shear stress on the vertical faces (or x-x axis) is taken as positive, whereas the shear stress on the horizontal faces (or y-y axis) is taken as negative.

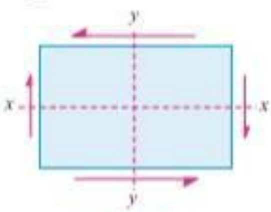

Fig. :1

Stresses on an Oblique Section of a Body Subjected to a Direct Stress in One Plane

Consider a rectangular body of uniform cross-sectional area and unit thickness subjected to a direct tensile stress along *x-x* axis as shown in Fig. 7.2 (*a*). Now let us consider an oblique section *AB*

inclined with the *x-x* axis (*i.e.*, with the line of action of the tensile stress on which we are required to find out the stresses as shown in the figure).

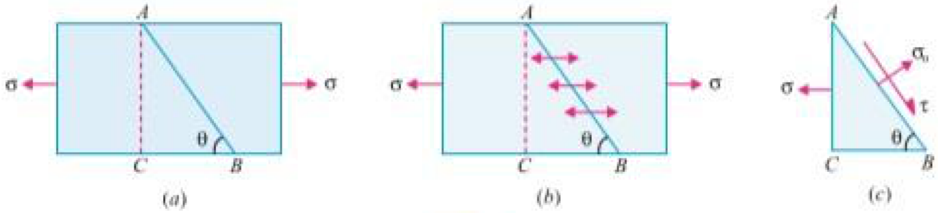

Fig. 7.2

Let σ = Tensile stress across the face *AC* and

θ = Angle, which the oblique section *AB* makes with *BC* *i.e.* with the *x-x* axis in the clockwise direction.

First of all, consider the equilibrium of an element or wedge *ABC* whose free body diagram is shown in fig 7.2 (*b*) and (*c*). We know that the horizontal force acting on the face *AC*,

$$P = \sigma \,.\, AC \;(\leftarrow)$$

Resolving the force perpendicular or normal to the section *AB*

$$P_n = P\sin\theta = \sigma \,.\, AC\sin\theta \qquad \text{....}(i)$$

and now resolving the force tangential to the section *AB*,

$$P_t = P\cos\theta = \sigma \,.\, AC\cos\theta \qquad \text{....}(ii)$$

We know that normal stress across the section *AB**,

$$\sigma_n = \frac{P_n}{AB} = \frac{\sigma AC\sin\theta}{AB} = \frac{\sigma . AC\sin\theta}{\frac{AC}{\sin\theta}} = \sigma\sin^2\theta$$

$$= \frac{\sigma}{2}(1-\cos 2\theta) = \frac{\sigma}{2} - \frac{\sigma}{2}\cos 2\theta \qquad \text{...}(iii)$$

and shear stress (*i.e.*, tangential stress) across the section *AB*,

$$\tau = \frac{P_t}{AB} = \frac{\sigma . AC\cos\theta}{AB} = \frac{\sigma . AC\cos\theta}{\frac{AC}{\sin\theta}} = \sigma\sin\theta\cos\theta$$

$$= \frac{\sigma}{2}\sin 2\theta \qquad \text{...}(iv)$$

It will be interesting to know from equation (*iii*) above that the normal stress across the section *AB* will be maximum, when $\sin^2\theta = 1$ or $\sin\theta = 1$ or $\theta = 90°$. Or in other words, the face *AC* will carry the maximum direct stress. Similarly, the shear stress across the section *AB* will be maximum when $\sin 2\theta = 1$ or $2\theta = 90°$ or $270°$. Or in other words, the shear stress will be maximum on the planes inclined at 45° and 135° with the line of action of the tensile stress. Therefore maximum shear stress when θ is equal to 45°,

$$\tau_{max} = \frac{\sigma}{2}\sin 90° = \frac{\sigma}{2}\times 1 = \frac{\sigma}{2}$$

and maximum shear stress, when θ is equal to 135°,

$$\tau_{max} = -\frac{\sigma}{2}\sin 270° = -\frac{\sigma}{2}(-1) = \frac{\sigma}{2}$$

It is thus obvious that the magnitudes of maximum shear stress is half of the tensile stress. Now the resultant stress may be found out from the relation :

$$\sigma_R = \sqrt{\sigma_n^2 + \tau^2}$$

Note : The planes of maximum and minimum normal stresses (*i.e.* principal planes) may also be found out by equating the shear stress to zero. This happens as the normal stress is either maximum or minimum on a plane having zero shear stress. Now equating the shear stress to zero,

$$\sigma \sin\theta \cos\theta = 0$$

It will be interesting to know that in the above equation either $\sin\theta$ is equal to zero or $\cos\theta$ is equal to zero. We know that if sin is zero, then θ is equal to 0°. Or in other words, the plane coincides with the line of action of the tensile stress. Similarly, if $\cos\theta$ is zero, then θ is equal to 90°. Or in other words, the plane is at right angles to the line of action of the tensile stress. Thus we see that there are two principal planes, at right angles to each other, one of them coincides with the line of action of the stress and the other at right angles to it.

1. *A wooden bar is subjected to a tensile stress of 5 MPa. What will be the values of normal and shear stresses across a section, which makes an angle of 25° with the direction of the tensile stress.*

Solution. Given : Tensile stress $(\sigma) = 5$ MPa and angle made by section with the direction of the tensile stress $(\theta) = 25°$.

Normal stress across the section

We know that normal stress across the section

$$\sigma_n = \frac{\sigma}{2} - \frac{\sigma}{2}\cos 2\theta = \frac{5}{2} - \frac{5}{2}\cos(2\times 25°)\text{ MPa}$$
$$= 2.5 - 2.5\cos 50° = 2.5 - (2.5\times 0.6428)\text{ MPa}$$
$$= 2.5 - 1.607 = 0.89\text{ MPa}$$ **Ans.**

Shear stress across the section

We also know that shear stress across the section,

$$\tau = \frac{\sigma}{2}\sin 2\theta = \frac{\sigma}{2}\sin(2\times 25°) = 2.5\sin 50°\text{ MPa}$$
$$= 2.5\times 0.766 = 1.915\text{ MPa}$$ **Ans.**

2. *Two wooden pieces 100 mm × 100 mm in cross-section are joined together along a line AB as shown in Fig. 7.3.*

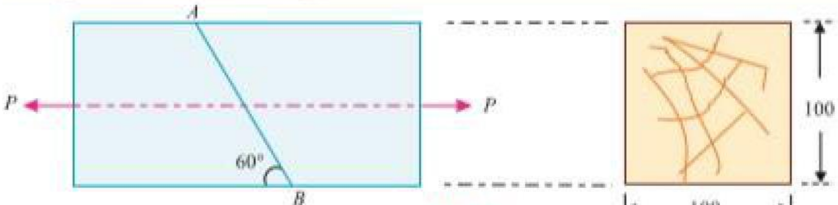

Fig. 7.3

Find the maximum force (P), which can be applied if the shear stress along the joint AB is 1.3 MPa.

Solution. Given : Section = 100 mm × 100 mm ; Angle made by section with the direction of tensile stress (θ) = 60° and permissible shear stress (τ) = 1.3 MPa = 1.3 N/mm^2.

Let $\quad \sigma$ = Safe tensile stress in the member

We know that cross- sectional area of the wooden member,

$$A = 100 \times 100 = 10\,000 \text{ mm}^2$$

and shear stress (τ),

$$1.3 = \frac{\sigma}{2}\sin 2\theta = \frac{\sigma}{2}\sin(2 \times 60°) = \frac{\sigma}{2}\sin 120° = \frac{\sigma}{2} \times 0.866$$

$$= 0.433\,\sigma$$

or
$$\sigma = \frac{1.3}{0.433} = 3.0 \text{ N/mm}^2$$

∴ Maximum axial force, which can be applied,

$$P = \sigma.A = 3.0 \times 10\,000 = 30\,000 \text{ N} = 30 \text{ kN} \qquad \textbf{Ans.}$$

$$P = \sigma.A = 3.0 \times 10\,000 = 30\,000 \text{ N} = 30 \text{ kN} \qquad \textbf{Ans.}$$

3. *A tension member is formed by connecting two wooden members 200 mm × 100 mm as shown in the figure given below:*

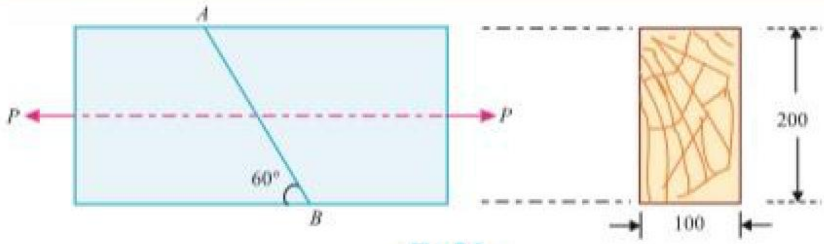

Fig. 7.4

Determine the safe value of the force (P), if permissible normal and shear stresses in the joint are 0.5 MPa and 1.25 MPa respectively.

Solution. Given : Section = 200 mm × 100 mm ; Angle made by section *AB* with the direction of the tensile stress (σ) = 60° ; Permissible normal stress (σ_n) = 0.5 MPa = 0.5 N/mm^2 and permissible shear stress (τ) = 1.25 MPa = 1.25 N/mm^2.

Let $\quad \sigma$ = Safe stress in the joint in N/mm^2.

We know that cross-sectional area of the member

$$A = 200 \times 100 = 20\,000 \text{ mm}^2$$

We also know that normal stress (σ_n),

$$0.5 = \frac{\sigma}{2} - \frac{\sigma}{2}\cos 2\theta = \frac{\sigma}{2} - \frac{\sigma}{2}\cos(2 \times 60^\circ)$$

$$= \frac{\sigma}{2} - \frac{\sigma}{2}\cos 120^\circ = \frac{\sigma}{2} - \frac{\sigma}{2}(-0.5) = 0.75\,\sigma$$

$$\therefore \quad \sigma = \frac{0.5}{0.75} = 0.67 \text{ N/mm}^2 \quad \ldots(i)$$

and shear stress (τ)

$$1.25 = \frac{\sigma}{2}\sin 2\theta = \frac{\sigma}{2}\sin(2 \times 60^\circ) = \frac{\sigma}{2}\sin 120^\circ = \frac{\sigma}{2} \times 0.866 = 0.433\sigma$$

$$\sigma = \frac{1.25}{0.433} = 2.89 \text{ N/mm}^2 \quad \ldots(ii)$$

From the above two values, we find that the safe stress is least of the two values, *i.e.* 0.67 N/mm^2. Therefore safe value of the force

$$P = \sigma \,.\, A = 0.67 \times 20\,000 = 13\,400 \text{ N} = 13.4 \text{ kN} \qquad \textbf{Ans.}$$

Stresses on an Oblique Section of a Body Subjected to Direct Stresses in Two Mutually Perpendicular Directions

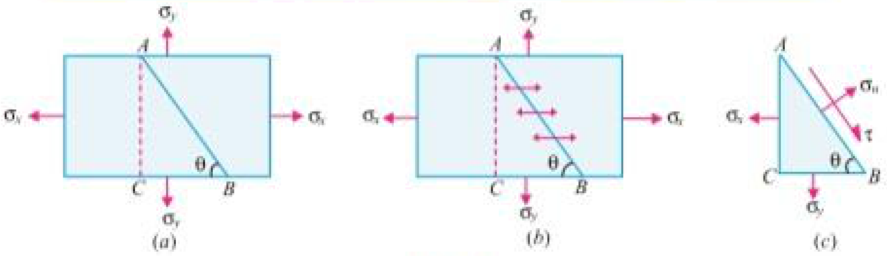

Fig. .5

Consider a rectangular body of uniform cross-sectional area and unit thickness subjected to direct tensile stresses in two mutually perpendicular directions along *x-x* and *y-y* axes as shown in Fig. 7.5. Now let us consider an oblique section *AB* inclined with *x-x* axis (*i.e.* with the line of action of the stress along *x-x* axis, termed as a major tensile stress on which we are required to find out the stresses as shown in the figure).

Let σ_x = Tensile stress along *x-x* axis (also termed as major tensile stress),

σ_y = Tensile stress along y-y axis (also termed as a minor tensile stress), and

θ = Angle which the oblique section *AB* makes with *x-x* axis in the clockwise direction.

First of all, consider the equilibrium of the wedge *ABC*. We know that horizontal force acting on the face *AC* (or *x-x* axis).

$$P_x = \sigma_x \,.\, AC\,(\leftarrow)$$

and vertical force acting on the face *BC* (or y-y axis),

$$P_y = \sigma_y \,.\, BC\,(\downarrow)$$

Resolving the forces perpendicular or normal to the section *AB*,

$$P_n = P_x \sin\theta + P_y \cos\theta = \sigma_x \,.\, AC \sin\theta + \sigma_y \,.\, BC \cos\theta \qquad ...(i)$$

and now resolving the forces tangential to the section *AB*,

$$P_t = P_x \cos\theta - P_y \sin\theta = \sigma_x \,.\, AC \cos\theta - \sigma_y \,.\, BC \sin\theta \qquad ...(ii)$$

We know that normal stress across the section *AB*,

$$\sigma_n = \frac{P_n}{AB} = \frac{\sigma_x . AC \sin\theta + \sigma_y \, BC \cos\theta}{AB}$$

$$= \frac{\sigma_x . AC \sin\theta}{AB} + \frac{\sigma_y . BC \cos\theta}{AB} = \frac{\sigma_x . AC \sin\theta}{\frac{AC}{\sin\theta}} + \frac{\sigma_y . BC \cos\theta}{\frac{BC}{\cos\theta}}$$

$$= \sigma_x \sin^2\theta + \sigma_y \,.\, \cos^2\theta = \frac{\sigma_x}{2}\,(1 - \cos 2\theta) + \frac{\sigma_y}{2}\,(1 + \cos 2\theta)$$

$$= \frac{\sigma_x}{2} - \frac{\sigma_x}{2}\cos 2\theta + \frac{\sigma_y}{2} + \frac{\sigma_y}{2}\cos 2\theta$$

$$= \frac{\sigma_x + \sigma_y}{2} - \frac{\sigma_x - \sigma_y}{2}\cos 2\theta \qquad ...(iii)$$

and shear stress (*i.e.*, tangential stress) across the section *AB*,

$$\tau = \frac{P_t}{AB} = \frac{\sigma_x . AC \cos\theta - \sigma_y . BC \sin\theta}{AB}$$

$$= \frac{\sigma_x . AC \cos\theta}{AB} - \frac{\sigma_y . BC \sin\theta}{AB} = \frac{\sigma_x . AC \cos\theta}{\frac{AC}{\sin\theta}} - \frac{\sigma_y . BC \sin\theta}{\frac{BC}{\cos\theta}}$$

$$= \sigma_x \,.\, \sin\theta\cos\theta - \sigma_y \sin\theta\cos\theta$$

$$= (\sigma_x - \sigma_y)\sin\theta\cos\theta = \frac{\sigma_x - \sigma_y}{2}\,\sin 2\theta \qquad ...(iv)$$

It will be interesting to know from equation (*iii*) the shear stress across the section *AB* will be maximum when $\sin 2\theta = 1$ or $2\theta = 90°$ or $\theta = 45°$. Therefore maximum shear stress,

$$\tau_{max} = \frac{\sigma_x - \sigma_y}{2}$$

Now the resultant stress may be found out from the relation :

$$\sigma_R = \sqrt{\sigma_n^2 + \tau^2}$$

4. *A point in a strained material is subjected to two mutually perpendicular tensile stresses of 200 MPa and 100 MPa. Determine the intensities of normal, shear and resultant stresses on a plane inclined at 30° with the axis of minor tensile stress.*

Solution. Given : Tensile stress along *x-x* axis (σ_x) =150 MPa ; Tensile stress along y-y axis (σ_y) = 100 MPa and angle made by plane with the axis of tensile stress $\theta = 30°$

Normal stress on the inclined plane

We know that normal stress on the inclined plane,

$$\sigma_n = \frac{\sigma_x + \sigma_y}{2} - \frac{\sigma_x - \sigma_y}{2}\cos 2\theta$$

$$= \frac{200+100}{2} - \frac{20-100}{2} \cos(2 \times 30°) \text{ MPa}$$

$$= 150 - (50 \times 0.5) = 125 \text{ MPa} \quad \textbf{Ans.}$$

Shear stress on the inclined plane

We know that shear stress on the inclined plane,

$$\tau = \frac{\sigma_x - \sigma_y}{2} \sin 2\theta = \frac{200-100}{2} \times \sin(2 \times 30°) \text{ MPa}$$

$$= 50 \sin 60° = 50 \times 0.866 = 43.3 \text{ MPa} \quad \textbf{Ans.}$$

Resultant stress on the inclined plane

We also know that resultant stress on the inclined plane,

$$\sigma_R = \sqrt{\sigma_n^2 + \tau^2} = \sqrt{(125)^2 + (43.3)^2} = 132.3 \text{ MPa} \quad \textbf{Ans.}$$

EXAMPLE 7.5. *The stresses at point of a machine component are 150 MPa and 50 MPa both tensile. Find the intensities of normal, shear and resultant stresses on a plane inclined at an angle of 55° with the axis of major tensile stress. Also find the magnitude of the maximum shear stress in the component.*

SOLUTION. Given : Tensile stress along *x-x* axis (σ_x) = 150 MPa ; Tensile stress along y-y axis (σ_y) = 50 MPa and angle made by the plane with the major tensile stress (θ) = 55°.

Normal stress on the inclined plane

We know that the normal stress on the inclined plane,

$$\sigma_n = \frac{\sigma_x + \sigma_y}{2} - \frac{\sigma_x - \sigma_y}{2} \cos 2\theta$$

$$= \frac{150+50}{2} - \frac{150-50}{2} \cos(2 \times 55°) \text{ MPa}$$

$$= 100 - 50 \cos 110° = 100 - 50(-0.342) \text{ MPa}$$

$$= 10 + 17.1 = 117.1 \text{ MPa} \quad \textbf{Ans.}$$

Shear stress on the inclined plane

We know that the shear stress on the inclined plane,

$$\tau = \frac{\sigma_x - \sigma_y}{2} \sin 2\theta = \frac{150-50}{2} \times \sin(2 \times 55°) \text{ MPa}$$

$$= 50 \sin 110° = 50 \times 0.9397 = 47 \text{ MPa} \quad \textbf{Ans.}$$

Resultant stress on the inclined plane

We know that resultant stress on the inclined plane,

$$\sigma_R = \sqrt{\sigma_n^2 + \tau^2} = \sqrt{(117.1)^2 + (47.0)^2} = 126.2 \text{ MPa} \quad \textbf{Ans.}$$

Maximum shear stress in the component

We also know that the magnitude of the maximum shear stress in the component,

$$\tau_{max} = \pm \frac{\sigma_x - \sigma_y}{2} = \pm \frac{150-50}{2} = \pm 50 \text{ MPa} \quad \textbf{Ans.}$$

.6. *The stresses at a point in a component are 100 MPa (tensile) and 50 MPa (compressive). Determine the magnitude of the normal and shear stresses on a plane inclined at an angle of 25° with tensile stress. Also determine the direction of the resultant stress and the magnitude of the maximum intensity of shear stress.*

Solution. Given : Tensile stress along *x-x* axis (σ_x) 100 MPa ; Compressive stress along *y-y* axis (σ_y) = –50 MPa (Minus sign due to compression) and angle made by the plane with tensile stress (θ) = 25°.

Normal stress on the inclined plane

We know that the normal stress on the inclined plane,

$$\sigma_n = \frac{\sigma_x + \sigma_y}{2} - \frac{\sigma_x - \sigma_y}{2}\cos 2\theta$$

$$= \frac{100 + (-50)}{2} - \frac{100 - (-50)}{2}\cos(2 \times 25°)\text{ MPa}$$

$$= 25 - 75\cos 50° = 25 - (75 \times 0.6428) = -23.21\text{ MPa}$$ **Ans.**

Shear stress on the inclined plane

We know that the shear stress on the inclined plane,

$$\tau = \frac{\sigma_x - \sigma_y}{2}\sin 2\theta = \frac{100 - (-50)}{2}\sin(2 \times 25°)\text{ MPa}$$

$$= 75\sin 50° = 75 \times 0.766 = 57.45\text{ MPa}$$ **Ans.**

Direction of the resultant stress

Let θ = Angle, which the resultant stress makes with *x-x* axis.

We know that $\tan\theta = \frac{\tau}{\sigma_n} = \frac{57.45}{-23.21} = -2.4752$ or $\theta = -68°$ **Ans.**

Maximum shear stress

We also know that magnitude of the maximum shear stress,

$$\tau_{max} = \pm\frac{\sigma_x - \sigma_y}{2} = \pm\frac{100 - (-50)}{2} = \pm 75\text{ MPa}$$ **Ans.**

Stresses on an Oblique Section of a Body Subjected to a Simple Shear stress

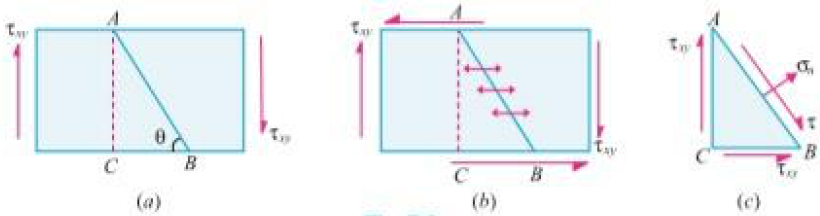

Fig. 7.6

Consider a rectangular body of uniform cross-sectional area and unit thickness subjected to a positive (*i.e.*, clockwise) shear stress along *x-x* axis as shown in Fig.7.6 (*a*). Now let us consider an oblique section *AB* inclined with *x-x* axis on which we are required to find out the stresses as shown in the figure 7.6 (*b*).

Let τ_{xy} = Positive (*i.e.*, clockwise) shear stress along *x-x* axis, and

θ = Angle , which the oblique section *AB* makes with *x-x* axis in the anticlockwise direction.

First of all, consider the equilibrium of the wedge *ABC*. We know that as per the principle of simple shear, the face *BC*, of the wedge will be subjected to an anticlockwise shear stress equal to τ_{xy} as shown in the Fig. 7.6 (*b*). We know that vertical force acting on the face *AC*,

$$P_1 = \tau_{xy} \cdot AC\ (\uparrow)$$

and horizontal force acting on the face BC,

$$P_2 = \tau_{xy} . BC (\rightarrow)$$

Resolving the forces perpendicular or normal to the AB,

$$P_n = P_1 \cos\theta + P_2 \sin\theta = \tau_{xy} . AC \cos\theta + \tau_{xy} . BC \sin\theta$$

and now resolving the forces tangential to the section AB,

$$P_t = P_2 \sin\theta - P_1 \cos\theta = \tau_{xy} . BC \sin\theta - \tau_{xy} . AC \cos\theta$$

We know that normal stress across the section AB,

$$\sigma_n = \frac{P_n}{AB} = \frac{\tau_{xy} . AC \cos\theta + \tau_{xy} . BC \sin\theta}{AB}$$

$$= \frac{\tau_{xy} . AC \cos\theta}{AB} + \frac{\tau_{xy} . BC \sin\theta}{AB}$$

$$= \frac{\tau_{xy} . AC \cos\theta}{\frac{AC}{\sin\theta}} + \frac{\tau_{xy} . BC \sin\theta}{\frac{BC}{\cos\theta}}$$

$$= \tau_{xy} . \sin\theta \cos\theta + \tau_{xy} . \sin\theta \cos\theta$$

$$= 2\tau_{xy} . \sin\theta \cos\theta = \tau_{xy} . \sin 2\theta$$

and shear stress (*i.e.* tangential stress) across the section AB

$$\tau = \frac{P_t}{AB} = \frac{\tau_{xy} . BC \sin\theta - \tau_{xy} . AC \cos\theta}{AB}$$

$$= \frac{\tau_{xy} . BC \sin\theta}{AB} - \frac{\tau_{xy} . AC \cos\theta}{AB} = \frac{\tau_{xy} . BC \sin\theta}{\frac{BC}{\sin\theta}} - \frac{\tau_{xy} . AC \cos\theta}{\frac{AC}{\cos\theta}}$$

$$= \tau_{xy} \sin^2\theta - \tau_{xy} \cos^2\theta$$

$$= \frac{\tau_{xy}}{2}(1 - \cos 2\theta) - \frac{\tau_{xy}}{2}(1 + \cos 2\theta)$$

$$= \frac{\tau_{xy}}{2} - \frac{\tau_{xy}}{2}\cos 2\theta - \frac{\tau_{xy}}{2} - \frac{\tau_{xy}}{2}\cos 2\theta$$

$$= -\tau_{xy} \cos 2\theta$$ ...(Minus sign means that normal stress is opposite to that across AC)

Now the planes of maximum and minimum normal stresses (*i.e.*, principal planes) may be found out by equating the shear stress to zero *i.e.*

$$-\tau_{xy} \cos 2\theta = 0$$

The above equation is possible only if $2\theta = 90°$ or $270°$ (because $\cos 90°$ or $\cos 270° = 0$) or in other words, $\theta = 45°$ or $135°$.

Stresses on an Oblique Section of a Body Subjected to a Direct Stress in One Plane and Accompanied by a Simple Shear Stress

Consider a rectangular body of uniform cross-sectional area and unit thickness subjected to a tensile stress along x-x axis accompanied by a positive (*i.e.* clockwise) shear stress along x-x axis as shown in Fig. 7.7 (*a*). Now let us consider an oblique section AB inclined with x-x axis on which we are required to find out the stresses as shown in the figure.

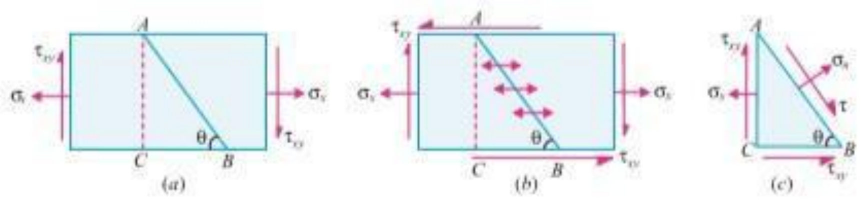

Fig. 7.7

Let σ_x = Tensile stress along x-x axis,

τ_{xy} = Positive (*i.e.* clockwise) shear stress along x-x axis, and

θ = Angle which the oblique section AB makes with x-x axis in clockwise direction.

First of all, consider the equilibrium of the wedge ABC. We know that as per the principle of simple shear, the face BC of the wedge will be subjected to an anticlockwise shear stress equal to τ_{xy} as shown in Fig. 7.7 (*b*). We know that horizontal force acting on the face AC,

$$P_x = \sigma_x . AC \; (\leftarrow) \qquad ...(i)$$

Similarly, vertical force acting on the face AC,

$$P_y = \tau_{xy} . AC \; (\uparrow) \qquad ...(ii)$$

and horizontal force acting on the face BC,

$$P = \tau_{xy} . BC \; (\rightarrow) \qquad ...(iii)$$

Resolving the forces perpendicular to the section AB,

$$P_n = P_x \sin\theta - P_y \cos\theta - P \sin\theta$$
$$= \sigma_x . AC \sin\theta - \tau_{xy} . AC \cos\theta - \tau_{xy} . BC \sin\theta$$

and now resolving the forces tangential to the section AB,

$$P_t = P_x \cos\theta + P_y \sin\theta - P \cos\theta$$
$$= \sigma_x . AC \cos\theta + \tau_{xy} . AC \sin\theta - \tau_{xy} . BC \cos\theta$$

We know that normal stress across the section AB,

$$\sigma_n = \frac{P_n}{AB} = \frac{\sigma_x . AC \sin\theta - \tau_{xy} . AC \cos\theta - \tau_{xy} . BC \sin\theta}{AB}$$

$$= \frac{\sigma_x . AC \sin\theta}{AB} - \frac{\tau_{xy} . AC \cos\theta}{AB} - \frac{\tau_{xy} . BC \sin\theta}{AB}$$

$$= \frac{\sigma_x . AC \sin\theta}{\frac{AC}{\sin\theta}} - \frac{\tau_{xy} . AC \cos\theta}{\frac{AC}{\sin\theta}} - \frac{\tau_{xy} . BC \sin\theta}{\frac{BC}{\cos\theta}}$$

$$= \sigma_x . \sin^2\theta - \tau_{xy} \sin\theta \cos\theta - \tau_{xy} \sin\theta \cos\theta$$

$$= \frac{\sigma_x}{2}(1 - \cos 2\theta) - 2\,\tau_{xy} \sin\theta \cos\theta$$

$$= \frac{\sigma_x}{2} - \frac{\sigma_x}{2}\cos 2\theta - \tau_{xy} \sin 2\theta \qquad ...(iv)$$

and shear stress (*i.e.*, tangential stress) across the section AB,

$$\tau = \frac{P_t}{AB} = \frac{\sigma_x . AC \cos\theta + \tau_{xy} . AC \sin\theta - \tau_{xy} . BC \cos\theta}{AB}$$

$$= \frac{\sigma_x . AC\cos\theta}{AB} + \frac{\tau_{xy}\, AC\sin\theta}{AB} - \frac{\tau_{xy} . BC\cos\theta}{AB}$$

$$= \frac{\sigma_x . AC\cos\theta}{\frac{AC}{\sin\theta}} + \frac{\tau_{xy}\, AC\sin\theta}{\frac{AC}{\sin\theta}} - \frac{\tau_{xy} . BC\cos\theta}{\frac{BC}{\cos\theta}}$$

$$= \sigma_x \sin\theta\cos\theta + \tau_{xy}\sin^2\theta - \tau_{xy}\cos^2\theta$$

$$= \frac{\sigma_x}{2}\sin 2\theta + \frac{\tau_{xy}}{2}(1-\cos 2\theta) - \frac{\tau_{xy}}{2}(1+\cos 2\theta)$$

$$= \frac{\sigma_x}{2}\sin 2\theta + \frac{\tau_{xy}}{2} - \frac{\tau_{xy}}{2}\cos 2\theta - \frac{\tau_{xy}}{2} - \frac{\tau_{xy}}{2}\cos 2\theta$$

$$= \frac{\sigma_x}{2}\sin 2\theta - \tau_{xy}\cos 2\theta \qquad ...(v)$$

Now the planes of maximum and minimum normal stresses (*i.e.*, principal planes) may be found out by equating the shear stress to zero *i.e.*, from the above equation, we find that the shear stress on any plane is a function of σ_x, τ_{xy} and θ. A little consideration will show that the values of σ_x and τ_{xy} are constant and thus the shear stress varies with the angle θ. Now let θ_p be the value of the angle for which the shear stress is zero.

$$\therefore \quad \frac{\sigma_x}{2}\sin 2\theta_p - \tau_{xy}\cos 2\theta_p = 0 \quad \text{or} \quad \frac{\sigma_x}{2}\sin 2\theta_p = \tau_{xy}\cos 2\theta_p$$

$$\therefore \quad \tan 2\theta_p = \frac{2\tau_{xy}}{\sigma_x}$$

From the above equation we find that the following two cases satisfy this condition as shown in Fig 7.8 (*a*) and (*b*)

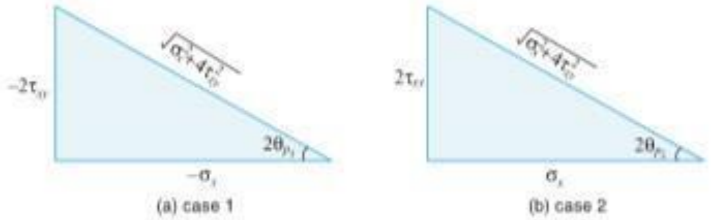

Fig. 7.8

Thus we find that these are two principal planes at right angles to each other, their inclination with *x-x* axis being θ_{p_1} and θ_{p_2}.

Now for case 1,

$$\sin 2\theta_{p_1} = \frac{-2\tau_{xy}}{\sqrt{\sigma_x^2 + 4\tau_{xy}^2}} \quad \text{and} \quad \cos 2\theta_{p_1} = \frac{-\sigma_x}{\sqrt{\sigma_x^2 + 4\tau_{xy}^2}}$$

Similarly for case 2,

$$\sin 2\theta_{p_2} = \frac{2\tau_{xy}}{\sqrt{\sigma_x^2 + 4\tau_{xy}^2}} \quad \text{and} \quad \cos 2\theta_{p_2} = \frac{\sigma_x}{\sqrt{\sigma_x^2 + 4\tau_{xy}^2}}$$

Now the values of principal stresses may be found out by substituting the above values of $2\theta_{p_1}$ and $2\theta_{p_2}$ in equation (*iv*).

Maximum principal stress,
$$\sigma_{p_1} = \frac{\sigma_x}{2} - \frac{\sigma_x}{2}\cos 2\theta - \tau_{xy}\sin 2\theta$$
$$= \frac{\sigma_x}{2} - \frac{\sigma_x}{2} \times \frac{-\sigma_x}{\sqrt{\sigma_x^2 + 4\tau_{xy}^2}} - \tau_{xy} \times \frac{-2\tau_{xy}}{\sqrt{\sigma_x^2 + 4\tau_{xy}^2}}$$
$$= \frac{\sigma_x}{2} + \frac{\sigma_x^2}{2\sqrt{\sigma_x^2 + 4\tau_{xy}^2}} + \frac{2\tau_{xy}^2}{\sqrt{\sigma_x^2 + 4\tau_{xy}^2}}$$
$$= \frac{\sigma_x}{2} + \frac{\sigma_x^2 + 4\tau_{xy}^2}{2\sqrt{\sigma_x^2 + 4\tau_{xy}^2}} = \frac{\sigma_x}{2} + \frac{\sqrt{\sigma_x^2 + 4\tau_{xy}^2}}{2}$$
$$= \frac{\sigma_x}{2} + \sqrt{\left(\frac{\sigma_x^2}{2}\right) + \tau_{xy}^2}$$

Minimum principal stress,
$$\sigma_{p_2} = \frac{\sigma_x}{2} - \frac{\sigma_x}{2}\cos 2\theta - \tau_{xy}\sin 2\theta$$
$$= \frac{\sigma_x}{2} - \frac{\sigma_x}{2} \times \frac{\sigma_x}{\sqrt{\sigma_x^2 + 4\tau_{xy}^2}} - \tau_{xy} \times \frac{2\tau_{xy}}{\sqrt{\sigma_x^2 + 4\tau_{xy}^2}}$$
$$= \frac{\sigma_x}{2} - \frac{\sigma_x^2}{2\sqrt{\sigma_x^2 + 4\tau_{xy}^2}} - \frac{2\tau_{xy}^2}{\sqrt{\sigma_x^2 + 4\tau_{xy}^2}}$$
$$= \frac{\sigma_x}{2} - \frac{\sqrt{\sigma_x^2 + 4\tau_{xy}^2}}{2} = \frac{\sigma_x}{2} - \frac{\sigma_x^2 + 4\tau_{xy}^2}{2\sqrt{\sigma_x^2 + 4\tau_{xy}^2}}$$
$$= \frac{\sigma_x}{2} - \sqrt{\left(\frac{\sigma_x}{2}\right)^2 + \tau_{xy}^2}$$

7. *A plane element in a body is subjected to a tensile stress of 100 MPa accompanied by a shear stress of 25 MPa. Find (i) the normal and shear stress on a plane inclined at an angle of 20° with the tensile stress and (ii) the maximum shear stress on the plane.*

SOLUTION. Given : Tensile stress along x-x axis $(\sigma_x) = 100$ MPa ; Shear stress $(\tau_{xy}) = 25$ MPa and angle made by plane with tensile stress $(\theta) = 20°$.

Normal and shear stresses on inclined section

We know that the normal stress on the plane,
$$\sigma_n = \frac{\sigma_x}{2} - \frac{\sigma_x}{2}\cos 2\theta - \tau_{xy}\sin 2\theta$$
$$= \frac{100}{2} - \frac{100}{2}\cos(2 \times 20°) - 25\sin(2 \times 20°) \text{ MPa}$$
$$= 50 - 50\cos 40° - 25\sin 40° \text{ MPa}$$
$$= 50 - (50 \times 0.766) - (25 \times 0.6428) \text{ MPa}$$
$$= 50 - 38.3 - 16.07 = -4.37 \text{ MPa} \quad \textbf{Ans.}$$

and shear stress on the plane,
$$\tau = \frac{\sigma_x}{2}\sin 2\theta - \tau_{xy}\cos 2\theta$$
$$= \frac{100}{2}\sin(2 \times 20°) - 25\cos(2 \times 20°) \text{ MPa}$$
$$= 50\sin 40° - 25\cos 40° \text{ MPa}$$
$$= (50 \times 0.6428) - (25 \times 0.766) \text{ MPa}$$
$$= 32.14 - 19.15 = 12.99 \text{ MPa} \quad \textbf{Ans.}$$

Maximum shear stress on the plane

We also know that maximum shear stress on the plane,
$$\tau_{max} = \sqrt{\left(\frac{\sigma_x}{2}\right)^2 + \tau_{xy}^2} = \sqrt{\left(\frac{100}{2}\right)^2 + (25)^2} = 55.9 \text{ MPa} \quad \textbf{Ans.}$$

8. *An element in a strained body is subjected to a tensile stress of 150 MPa and a shear stress of 50 MPa tending to rotate the element in an anticlockwise direction. Find (i) the magnitude of the normal and shear stresses on a section inclined at 40° with the tensile stress; and (ii) the magnitude and direction of maximum shear stress that can exist on the element.*

SOLUTION. Given : Tensile stress along horizontal x-x axis $(\sigma_x) = 150$ MPa ; Shear stress $(\tau_{xy}) - 50$ MPa (Minus sign due to anticlockwise) and angle made by section with the tensile stress $(\theta) = 40°$.

Normal and Shear stress on the inclined section

We know that magnitude of the normal stress on the section,

$$\sigma_n = \frac{\sigma_x}{2} - \frac{\sigma_x}{2}\cos 2\theta - \tau_{xy}\sin 2\theta$$
$$= \frac{150}{2} - \frac{150}{2}\cos(2\times 40°) - (-50)\sin(2\times 40°) \text{ MPa}$$
$$= 75 - (75\times 0.1736) + (50\times 0.9848) \text{ MPa}$$
$$= 75 - 13.02 + 49.24 = 111.22 \text{ MPa} \quad \textbf{Ans.}$$

and shear stress on the section

$$\tau = \frac{\sigma_x}{2}\sin 2\theta - \tau_{xy}\cos 2\theta$$
$$= \frac{150}{2}\sin(2\times 40°) - (-50)\cos(2\times 40°) \text{ MPa}$$
$$= (75\times 0.9848) + (50\times 0.1736) \text{ MPa}$$
$$= 73.86 + 8.68 = 82.54 \text{ MPa} \quad \textbf{Ans.}$$

(ii) Maximum shear stress and its direction that can exist on the element

We know that magnitude of the maximum shear stress.

$$\tau_{max} = \pm\sqrt{\left(\frac{\sigma_x}{2}\right)^2 + \tau_{xy}^2} = \pm\sqrt{\left(\frac{150}{2}\right)^2 + (-50)^2} = \pm 90.14 \text{ MPa} \quad \textbf{Ans.}$$

Let θ_s = Angle which plane of maximum shear stress makes with x-x axis.

We know that, $$\tan 2\theta_s = \frac{\sigma_x}{2\tau_{xy}} = \frac{150}{2\times 50} = 1.5 \quad \text{or} \quad 2\theta_s = 56.3°$$

$$\therefore \quad \theta_s = 28.15° \quad \text{or} \quad 118.15° \quad \textbf{Ans.}$$

9. *An element in a strained body is subjected to a compressive stress of 200 MPa and a clockwise shear stress of 50 MPa on the same plane. Calculate the values of normal and shear stresses on a plane inclined at 35° with the compressive stress. Also calculate the value of maximum shear stress in the element.*

Solution. Given : Compressive stress along horizontal x-x axis (σ_x) = – 200 MPa (Minus sign due to compressive stress) ; Shear stress (τ_{xy}) = 50 MPa and angle made by the plane with the compressive stress (θ) = 35°

Normal and shear stresses across inclined section

We know that normal stress on the plane,

$$\begin{aligned}\sigma_n &= \frac{\sigma_x}{2} - \frac{\sigma_x}{2}\cos 2\theta - \tau_{xy}\sin 2\theta \\ &= \frac{-200}{2} - \frac{-200}{2}\cos(2\times 35°) - 50\sin(2\times 35°)\ \text{MPa} \\ &= -100 + (100\times 0.342) - (50\times 0.94)\ \text{MPa} \\ &= -100 + 34.2 - 46.9 = -112.9\ \text{MPa} \quad \textbf{Ans.}\end{aligned}$$

and shear stress on the plane,

$$\begin{aligned}\tau &= \frac{\sigma_x}{2}\sin 2\theta - \tau_{xy}\cos 2\theta \\ &= \frac{-200}{2}\sin(2\times 35°) - 50\cos(2\times 35°)\ \text{MPa} \\ &= (-100\times 0.9397) - (50\times 0.342)\ \text{MPa} \\ &= -93.97 - 17.1 = -111.07\ \text{MPa} \quad \textbf{Ans.}\end{aligned}$$

Maximum shear stress in the element

We also know that value of maximum shear stress in the element,

$$\tau_{max} = \sqrt{\left(\frac{\sigma_x}{2}\right)^2 + \tau_{xy}^2} = \sqrt{\left(\frac{-200}{2}\right)^2 + (50)^2} = 111.8\ \text{MPa} \quad \textbf{Ans.}$$

. Stresses on an Oblique Section of a Body Subjected to Direct Stresses in Two Mutually Perpendicular Directions Accompanied by a Simple Shear Stress

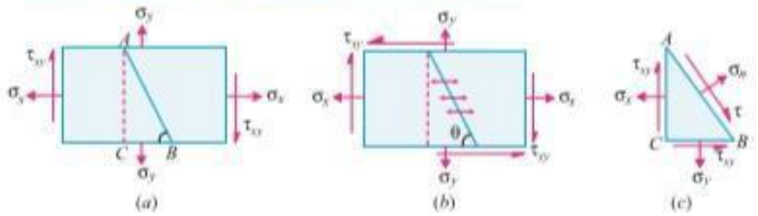

Fig. 7.9

Consider a rectangular body of uniform cross-sectional area and unit thickness subjected to tensile stresses along *x-x* and y-y axes and accompanied by a positive (*i.e.*, clockwise) shear stress along *x-x* axis as shown in Fig.7.9 (*b*). Now let us consider an oblique section *AB* inclined with *x-x* axis on which we are required to find out the stresses as shown in the figure.

Let σ_x = Tensile stress along *x-x* axis,

σ_y = Tensile stress along y-y axis,

τ_{xy} = Positive (*i.e.* clockwise) shear stress along *x-x* axis, and

θ = Angle, which the oblique section *AB* makes with *x-x* axis in an anticlockwise direction.

First of all, consider the equilibrium of the wedge *ABC*. We know that as per the principle of simple shear, the face *BC* of the wedge will be subjected to an anticlockwise shear stress equal to τ_{xy}

as shown in Fig. 7.9 (*b*). We know that horizontal force acting on the face *AC*,

$$P_1 = \sigma_x . AC \, (\leftarrow) \quad \text{...(i)}$$

and vertical force acting on the face *AC*,

$$P_2 = \tau_{xy} . AC \, (\uparrow) \quad \text{...(ii)}$$

Similarly, vertical force acting on the face *BC*,

$$P_3 = \sigma_y . BC \, (\downarrow) \quad \text{...(iii)}$$

and horizontal force on the face *BC*,

$$P_4 = \tau_{xy} . BC \, (\rightarrow) \quad \text{...(iv)}$$

Now resolving the forces perpendicular to the section *AB*,

$$P_n = P_1 \sin\theta - P_2 \cos\theta + P_3 \cos\theta - P_4 \sin\theta$$
$$= \sigma_x . AC \sin\theta - \tau_{xy} . AC \cos\theta + \sigma_y . BC \cos\theta - \tau_{xy} . BC \sin\theta$$

and now resolving the forces tangential to *AB*,

$$P_t = P_1 \cos\theta + P_2 \sin\theta - P_3 \sin\theta - P_4 \cos\theta$$
$$= \sigma_x . AC \cos\theta + \tau_{xy} . AC \sin\theta - \sigma_y . BC \sin\theta - \tau_{xy} . BC \cos\theta$$

Normal Stress (across the inclined section *AB*)

$$\sigma_n = \frac{P_n}{AB} = \frac{\sigma_x . AC \sin\theta - \tau_{xy} . AC \cos\theta + \sigma_y . BC \cos\theta - \tau_{xy} . BC \sin\theta}{AB}$$

$$= \frac{\sigma_x . AC \sin\theta}{AB} - \frac{\tau_{xy} . AC \cos\theta}{AB} + \frac{\sigma_y . BC \cos\theta}{AB} - \frac{\tau_{xy} . BC \sin\theta}{AB}$$

$$= \frac{\sigma_x . AC \sin\theta}{\frac{AC}{\sin\theta}} - \frac{\tau_{xy} . AC \cos\theta}{\frac{AC}{\sin\theta}} + \frac{\sigma_y . BC \cos\theta}{\frac{BC}{\cos\theta}} - \frac{\tau_{xy} . BC \sin\theta}{\frac{BC}{\cos\theta}}$$

$$= \sigma_x . \sin^2\theta - \tau_{xy} \sin\theta \cos\theta + \sigma_y . \cos^2\theta - \tau_{xy} . \sin\theta \cos\theta$$

$$= \frac{\sigma_x}{2} (1 - \cos 2\theta) + \frac{\sigma_y}{2} (1 + \cos 2\theta) - 2\,\tau_{xy} . \sin\theta \cos\theta$$

$$= \frac{\sigma_x}{2} - \frac{\sigma_x}{2} \cos 2\theta + \frac{\sigma_y}{2} + \frac{\sigma_y}{2} \cos 2\theta - \tau_{xy} \sin 2\theta$$

or

$$\sigma_n = \frac{\sigma_x + \sigma_y}{2} - \frac{\sigma_x - \sigma_y}{2} \cos 2\theta - \tau_{xy} \sin 2\theta \quad \text{...(v)}$$

Shear Stress or Tangential Stress (across inclined the section *AB*)

$$\tau = \frac{P_t}{AB} = \frac{\sigma_x . AC \cos\theta + \tau_{xy} . AC \sin\theta - \sigma_y . BC \sin\theta - \tau_{xy} BC \cos\theta}{AB}$$

$$= \frac{\sigma_x . AC \cos\theta}{AB} + \frac{\tau_{xy} . AC \sin\theta}{AB} - \frac{\sigma_y . BC \sin\theta}{AB} - \frac{\tau_{xy} BC \cos\theta}{AB}$$

$$= \frac{\sigma_x . AC \cos\theta}{\frac{AC}{\sin\theta}} + \frac{\tau_{xy} . AC \sin\theta}{\frac{AC}{\sin\theta}} - \frac{\sigma_y . BC \sin\theta}{\frac{BC}{\cos\theta}} - \frac{\tau_{xy} . BC \cos\theta}{\frac{BC}{\cos\theta}}$$

$$= \sigma_x \sin\theta \cos\theta + \tau_{xy} \sin^2\theta - \sigma_y \sin\theta \cos\theta - \tau_{xy} \cos^2\theta$$

$$= (\sigma_x - \sigma_y) \sin\theta \cos\theta + \frac{\tau_{xy}}{2} (1 - \cos 2\theta) - \frac{\tau_{xy}}{2} (1 + \cos 2\theta)$$

or

$$\tau = \frac{\sigma_x - \sigma_y}{2} \sin 2\theta - \tau_{xy} \cos 2\theta \quad \text{...(vi)}$$

Now the planes of maximum and minimum normal stresses (*i.e.* principal planes) may be found out by equating the shear stress to zero. From the above equations, we find that the shear stress to any plane is a function of σ_y, σ_x, τ_{xy} and θ. A little consideration will show that the values of σ_y, σ_x and τ_{xy} are constant and thus the shear stress varies in the angle θ. Now let θ_p be the value of the angle for which the shear stress is zero.

$$\therefore \quad \frac{\sigma_x - \sigma_y}{2} \sin 2\theta_p - \tau_{xy} \cos 2\theta_p = 0$$

or $$\frac{\sigma_x - \sigma_y}{2} \sin 2\theta_p = \tau_{xy} \cos 2\theta_p \qquad \text{or} \qquad \tan 2\theta_p = \frac{2\tau_{xy}}{\sigma_x - \sigma_y}$$

From the above equation, we find that the following two cases satisfy this condition as shown in Fig 7.10 (*a*) and (*b*).

(*a*) Case 1 (*b*) Case 2

Fig. 7.10

Thus we find that there are two principal planes, at right angles to each other, their inclinations with *x-x* axis being θ_{p_1} and θ_{p_2}.

Now for case 1,

$$\sin 2\theta_{p_1} = \frac{-2\tau_{xy}}{\sqrt{(\sigma_x-\sigma_y)^2+4\tau_{xy}^2}} \quad \text{and} \quad \cos 2\theta_{p_1} = \frac{-(\sigma_x-\sigma_y)}{\sqrt{(\sigma_x-\sigma_y)^2+4\tau_{xy}^2}}$$

Similarly for case 2,

$$\sin 2\theta_{p_2} = \frac{2\tau_{xy}}{\sqrt{(\sigma_x-\sigma_y)^2+4\tau_{xy}^2}} \quad \text{and} \quad \cos 2\theta_{p_2} = \frac{(\sigma_x-\sigma_y)}{\sqrt{(\sigma_x-\sigma_y)^2+4\tau_{xy}^2}}$$

Now the values of principal stresses may be found out by substituting the above values of $2\theta_{p_1}$ and $2\theta_{p_2}$ in equation (*v*).

Maximum Principal Stress,

$$\sigma_{p_1} = \frac{\sigma_x+\sigma_y}{2} - \frac{\sigma_x-\sigma_y}{2}\cos 2\theta - \tau_{xy}\sin 2\theta$$

$$= \frac{\sigma_x+\sigma_y}{2} - \left(\frac{\sigma_x-\sigma_y}{2} \times \frac{-(\sigma_x-\sigma_y)}{\sqrt{(\sigma_x-\sigma_y^2)+4\tau_{xy}^2}}\right) - \left(\tau_{xy} \times \frac{-2\tau_{xy}}{\sqrt{(\sigma_x-\sigma_y)^2+4\tau_{xy}^2}}\right)$$

$$= \frac{\sigma_x+\sigma_y}{2} + \frac{(\sigma_x-\sigma_y)^2+4\tau_{xy}^2}{2\sqrt{\sigma_x-\sigma_y)^2+4\tau_{xy}^2}} = \frac{\sigma_x+\sigma_y}{2} + \frac{\sqrt{(\sigma_x-\sigma_y)^2+4\tau_{xy}^2}}{2}$$

or $$\sigma_{p_1} = \frac{\sigma_x+\sigma_y}{2} + \sqrt{\left(\frac{\sigma_x-\sigma_y}{2}\right)^2+\tau_{xy}^2}$$

Minimum Principal Stress

$$\sigma_{p2} = \frac{\sigma_x+\sigma_y}{2} - \frac{(\sigma_x-\sigma_y)}{2}\cos 2\theta - \tau_{xy}\sin 2\theta$$

$$= \frac{\sigma_x+\sigma_y}{2} - \left(\frac{\sigma_x-\sigma_y}{2}\times\frac{(\sigma_x-\sigma_y)}{\sqrt{(\sigma_x-\sigma_y)^2+4\tau_{xy}^2}}\right) - \left(\tau_{xy}\times\frac{2\tau_{xy}}{\sqrt{(\sigma_x-\sigma_y)^2+4\tau_{xy}^2}}\right)$$

$$= \frac{\sigma_x+\sigma_y}{2} - \frac{(\sigma_x-\sigma_y)^2+4\tau_{xy}^2}{2\sqrt{(\sigma_x-\sigma_y)^2+4\tau_{xy}^2}} = \frac{\sigma_x-\sigma_y}{2} - \frac{\sqrt{(\sigma_x-\sigma_y)^2+4\tau_{xy}^2}}{2}$$

or
$$\sigma_{p_2} = \frac{\sigma_x+\sigma_y}{2} - \sqrt{\left(\frac{\sigma_x-\sigma_y}{2}\right)^2+\tau_{xy}^2}$$

10. *A point is subjected to a tensile stress of 250 MPa in the horizontal direction and another tensile stress of 100 MPa in the vertical direction. The point is also subjected to a simple shear stress of 25 MPa, such that when it is associated with the major tensile stress, it tends to rotate the element in the clockwise direction. What is the magnitude of the normal and shear stresses on a section inclined at an angle of 20° with the major tensile stress?*

SOLUTION. Given : Tensile stress in horizontal *x-x* direction (σ_x) = 250 MPa ; Tensile stress vertical *y-y* direction (σ_y) = 100 MPa ; Shear stress (τ_{xy}) = 25 MPa and angle made by section wi the major tensile stress (θ) = 20°.

Magnitude of normal stress

We know that magnitude of normal stress,

$$\sigma_n = \frac{\sigma_x+\sigma_y}{2} - \frac{\sigma_x-\sigma_y}{2}\cos 2\theta - \tau_{xy}\sin 2\theta$$
$$= \frac{250+100}{2} - \frac{250-100}{2}\cos(2\times 20°) - 25\sin(2\times 20°)$$
$$= 175 - 75\cos 40° - 25\sin 40° \text{ MPa}$$
$$= 175 - (75\times 0.766) - (25\times 0.6428) \text{ MPa}$$
$$= 175 - 57.45 - 16.07 = 101.48 \text{ MPa} \quad \textbf{Ans.}$$

Magnitude of shear stress

We also know that magnitude of shear stress,

$$\tau = \frac{\sigma_x-\sigma_y}{2}\sin 2\theta - \tau_{xy}\cos 2\theta$$
$$= \frac{250-100}{2}\sin(2\times 20°) - 25\cos(2\times 20°)$$
$$= 75\sin 40° - 25\cos 40° \text{ MPa}$$
$$= (75\times 0.6428) - (25\times 0.766) \text{ MPa}$$
$$= 48.21 - 19.15 = 29.06 \text{ MPa} \quad \textbf{Ans.}$$

11. *A plane element in a boiler is subjected to tensile stresses of 400 MPa on one plane and 150 MPa on the other at right angles to the former. Each of the above stresses is accompanied by a shear stress of 100 MPa such that when associated with the minor tensile stress tends to rotate the element in anticlockwise direction. Find*

(a) *Principal stresses and their directions.*

(b) *Maximum shearing stresses and the directions of the plane on which they act.*

SOLUTION. Given : Tensile stress along *x-x* axis $(\sigma_x) = 400$ MPa ; Tensile stress along *y-y* axis $(\sigma_y) = 150$ MPa and shear stress $(\tau_{xy}) = -100$ MPa (Minus sign due to anticlockwise on *x-x* direction).

(a) Principal stresses and their directions

We know that maximum principal stress,

$$\sigma_{max} = \frac{\sigma_x + \sigma_y}{2} + \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

$$= \frac{400+150}{2} + \sqrt{\left(\frac{400-150}{2}\right)^2 + (-100)^2} \text{ MPa}$$

$$= 275 + 160.1 = 435.1 \text{ MPa} \quad \textbf{Ans.}$$

and minimum principal stress,

$$\sigma_{min} = \frac{\sigma_x + \sigma_y}{2} - \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

$$= \frac{400+150}{2} - \sqrt{\left(\frac{400-150}{2}\right)^2 + (-100)^2} \text{ MPa}$$

$$= 275 - 160.1 = 114.9 \text{ MPa} \quad \textbf{Ans.}$$

Let θ_p = Angle which plane of principal stress makes with *x-x* axis.

We know that, $$\tan 2\theta_p = \frac{2\tau_{xy}}{\sigma_x - \sigma_y} = \frac{2 \times 100}{400-150} = 0.8 \quad \text{or} \quad 2\theta_p = 38.66^\circ$$

$$\therefore \quad \theta_p = 19.33^\circ \quad \text{or} \quad 109.33^\circ \quad \textbf{Ans.}$$

(b) Maximum shearing stresses and their directions

We also know that maximum shearing stress

$$\tau_{max} = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2} = \sqrt{\left(\frac{400-150}{2}\right)^2 + (-100)^2}$$

$$= 160.1 \text{ MPa} \quad \textbf{Ans.}$$

Let θ_s = Angle which plane of maximum shearing stress makes with *x-x* axis.

We know that, $$\tan 2\theta_s = \frac{\sigma_x - \sigma_y}{2\tau_{xy}} = \frac{400-150}{2 \times 100} = 1.25 \quad \text{or} \quad 2\theta_s = 51.34^\circ$$

$$\theta_s = 25.67^\circ \quad \text{or} \quad 115.67^\circ \quad \textbf{Ans.}$$

12. *A point in a strained material is subjected to the stresses as shown in Fig. 7.11.*

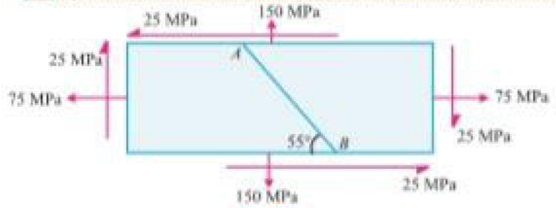

Fig. 7.11

Find graphically, or otherwise, the normal and shear stresses on the section AB.

Solution. Given : Tensile stress along horizontal *x-x* axis (σ_x) = 75 MPa ; Tensile stress along vertical y-y axis (σ_y) = 150 MPa ; Shear stress (τ_{xy}) = 25 MPa and angle made by section with the horizontal direction (θ) = 55°.

Normal stress on the section AB

We know that normal stress on the section *AB*,

$$\sigma_n = \frac{\sigma_x - \sigma_y}{2} - \frac{\sigma_x - \sigma_y}{2}\cos 2\theta - \tau_{xy}\sin 2\theta$$

$$= \frac{75+150}{2} - \frac{75-150}{2}\cos(2\times 55°) - 25\sin(2\times 55°)\text{ MPa}$$

$$= 112.5 + 37.5\cos 110° - 25\sin 110°\text{ MPa}$$

$$= 11.25 + 37.5\times(-0.342) - (25\times 0.9397)\text{ MPa}$$

$$= 112.5 - 12.83 - 23.49 = 76.18\text{ MPa} \quad \textbf{Ans.}$$

Shear stress on the section AB

We also know that shear stress on the section *AB*.

$$\tau = \frac{\sigma_x - \sigma_y}{2}\sin 2\theta - \tau_{xy}\cos 2\theta$$

$$= \frac{75-150}{2}\sin(2\times 55°) - 25\cos(2\times 55°)\text{ MPa}$$

$$= -37.5\sin 110° - 25\cos 110°\text{ MPa}$$

$$= -37.5\times 0.9397 - 25\times(-0.342)\text{ MPa}$$

$$= -35.24 + 8.55 = -26.69\text{ MPa} \quad \textbf{Ans.}$$

$$= \frac{-300-200}{2}\sin(2\times 30°) - [-100\cos(2\times 30°)]\text{ MPa}$$

$$= -250\sin 60° + 100\cos 60°\text{ MPa}$$

$$= -250\times 0.866 + 100\times 0.5\text{ MPa}$$

$$= -216.5 + 50 = -166.5\text{ MPa} \quad \textbf{Ans.}$$

EXAMPLE 7.14. *A machine component is subjected to the stresses as shown in the figure given below :*

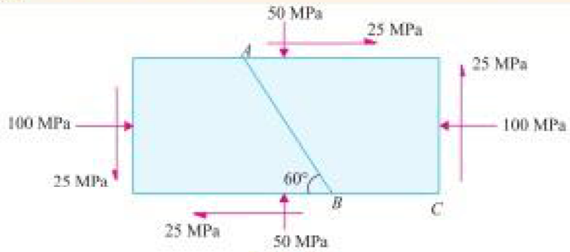

Fig. 7.12

Find the normal and shearing stresses on the section AB inclined at an angle of 60° with x-x axis. Also find the resultant stress on the section.

SOLUTION. Given : Compressive stress along horizontal *x-x* axis (σ_x) = – 100 MPa (Minus sign due to compressive stress) ; Compressive stress along vertical *y-y* axis (σ_y) = – 50 MPa (Minus sign due to compressive stress) ; Shear stress (τ_{xy}) = – 25 MPa (Minus sign due to anticlockwise on *x-x* axis) and angle made by section *AB* with *x-x* axis (θ) = 60°.

Normal stress on the section AB

We know that normal stress on the section *AB*,

$$\sigma_n = \frac{\sigma_x + \sigma_y}{2} - \frac{\sigma_x - \sigma_y}{2}\cos 2\theta - \tau_{xy}\sin 2\theta$$

$$= \frac{-100+(-50)}{2} - \frac{-100-(-50)}{2}\cos(2\times 60°) - [-25\sin(2\times 60°)]$$

$$= -75 + 25\cos 120° + 25\sin 120° \text{ MPa}$$

$$= -75 + [25\times(-0.5)] + (25\times 0.866) \text{ MPa}$$

$$= -75 - 12.5 + 21.65 = -65.85 \text{ MPa}$$ **Ans.**

Shearing stress on the section AB

We know that shearing stress on the section *AB*,

$$\tau = \frac{\sigma_x - \sigma_y}{2}\sin 2\theta - \tau_{xy}\cos 2\theta$$

$$= \frac{-100-(-50)}{2}\sin(2\times 60°) - [-25\cos(2\times 60°)]$$

$$= -25\sin 120° + 25\cos 120° = -25\times 0.866 + [25\times(-0.5)] \text{ MPa}$$

$$= -21.65 - 12.5 = -34.15 \text{ MPa}$$ **Ans.**

Resultant stress on the section AB

We also know that resultant stress on the section *AB*,

$$\sigma_R = \sqrt{\sigma_n^2 + \tau^2} = \sqrt{(-65.85)^2 + (-34.15)^2} = 74.2 \text{ MPa}$$ **Ans.**