Construction of Analytic function:

Milne's Thomson method

i). To find $f(z)$, when u is given

$$f(z) = \int [\phi_1(z,0) - i\phi_2(z,0)]\,dz$$

where $\phi_1(z,0) = \left(\frac{\partial u}{\partial x}\right)_{(z,0)}$ and

$$\phi_2(z,0) = \left(\frac{\partial u}{\partial y}\right)_{(z,0)}$$

ii). To find $f(z)$, when v is given

$$f(z) = \int [\phi_1(z,0) + i\phi_2(z,0)]\,dz$$

where $\phi_1(z,0) = \left(\frac{\partial v}{\partial y}\right)_{(z,0)}$ and

$$\phi_2(z,0) = \left(\frac{\partial v}{\partial x}\right)_{(z,0)}$$

iii). If $u-v$ or $u+v$ is given, then to find $f(z)$

Take $f(z) = u+iv$

$if(z) = iu - v$

1) Find the analytic function $f(z)$ whose real part is $u = 3x^2y + 2x^2 - y^3 - 2y^2$

Soln.

Given $u = 3x^2y + 2x^2 - y^3 - 2y^2$

$$\frac{\partial u}{\partial x} = 6xy + 4x$$

$$\phi_1(z,0) = \left(\frac{\partial u}{\partial x}\right)_{(z,0)} = 4z$$

$$\frac{\partial u}{\partial y} = 3x^2 - 3y^2 - 4y$$

$$\phi_2(z,0) = \left(\frac{\partial u}{\partial y}\right)_{(z,0)} = 3z^2$$

By Milne Thomson method,

$$f(z) = \int [\phi_1(z,0) - i\,\phi_2(z,0)]\,dz$$

$$= \int [4z - i\,3z^2]\,dz$$

$$= \frac{4z^2}{2} - i\,\frac{3z^3}{3} + c$$

$$f(z) = 2z^2 - iz^3 + c$$

HW $x^3 - 3xy^2 + 3x^2 - 3y^2$

Ans: $z^3 + 3z^2 + c$

2]. Prove that $v = e^{-x}(x\cos y + y\sin y)$ is harmonic and determine analytic function $f(z)$.

Soln.

Given $v = e^{-x}x\cos y + e^{-x}y\sin y$

$$\frac{\partial v}{\partial x} = [e^{-x} + x(-1)e^{-x}]\cos y + (-1)e^{-x}\,y\sin y$$

$$= e^{-x}\cos y - xe^{-x}\cos y - e^{-x}y\sin y$$

$$\frac{\partial^2 v}{\partial x^2} = -e^{-x}\cos y - [x(-e^{-x}) + e^{-x}]\cos y + e^{-x}y\sin y$$

$$= -e^{-x}\cos y + xe^{-x}\cos y - e^{-x}\cos y + e^{-x}y\sin y$$

$$\frac{\partial v}{\partial y} = e^{-x}x[-\sin y] + e^{-x}[y\cos y + \sin y]$$

$$= -xe^{-x}\sin y + e^{-x}y\cos y + e^{-x}\sin y$$

$$\frac{\partial^2 v}{\partial y^2} = -xe^{-x}\cos y + e^{-x}[y(-\sin y) + \cos y] + e^{-x}\cos y$$

$$= -xe^{-x}\cos y - y\sin y\,e^{-x} + e^{-x}\cos y + e^{-x}\cos y$$

$$= -xe^{-x}\cos y - e^{-x}y\sin y + 2e^{-x}\cos y$$

$$\therefore \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}$$

$$= -2e^{-x}\cos y + xe^{-x}\cos y + e^{-x}y\sin y$$
$$\quad - xe^{-x}\cos y - e^{-x}y\sin y + 2e^{-x}\cos y$$

$$= 0$$

Hence v satisfies laplace eqn.

$\therefore$ v is harmonic.

By Milne's Thomson method,

$$f(z) = \int [\phi_1(z,0) + i\,\phi_2(z,0)]\,dz$$

where $\phi_1(z,0) = \left(\frac{\partial v}{\partial y}\right)_{(z,0)}$

$$= [-xe^{-x}\sin y + e^{-x}y\cos y + e^{-x}\sin y]_{(z,0)}$$

$$\phi_1(z,0) = 0$$

$$\phi_2(z,0) = \left(\frac{\partial v}{\partial x}\right)_{(z,0)}$$

$$= [e^{-x}\cos y - xe^{-x}\cos y - e^{-x}y\sin y]_{(z,0)}$$

$$\phi_2(z,0) = e^{-z} - ze^{-z}$$

$$\therefore f(z) = \int [0 + i(e^{-z} - ze^{-z})]\,dz$$

$$= i\left[\int e^{-z}\,dz - \int ze^{-z}\,dz\right]$$

$$= i\left[-e^{-z} - (-ze^{-z} - e^{-z})\right] + c$$

$$= i\left[-e^{-z} + ze^{-z} + e^{-z}\right] + c$$

$$f(z) = ize^{-z} + c$$

Hw $v = e^{2x}(y\cos 2y + x\sin 2y)$

$f(z) = ?$

$v = x^2 - y^2 + 2x$

By Milne's Thomson method,

$$F(z) = \int [\phi_1(z,0) - i\,\phi_2(z,0)]\,dz$$

$$= \int (e^z + i e^z)\,dz$$

$$= (1+i)\int e^z\,dz$$

$$(1+i)\,f(z) = (1+i)\,e^z + c$$

$$f(z) = e^z + C$$

5]. If $f(z) = u+iv$ is analytic, find $f(z)$ given that $u+v = \dfrac{\sin 2x}{\cosh 2y - \cos 2x}$

Soln. Let $f(z) = u + iv \longrightarrow (1)$

$$i f(z) = iu - v \longrightarrow (2)$$

$$(1)+(2) \Rightarrow (1+i)\,f(z) = u + iv + iu - v$$

$$= (u-v) + i(u+v)$$

$$F(z) = U + iV$$

Here $F(z) = (1+i)\,f(z)$

$U = u - v$

$V = u + v$

Given $V = u + v = \dfrac{\sin 2x}{\cosh 2y - \cos 2x}$

$$\frac{\partial V}{\partial x} = \frac{(\cosh 2y - \cos 2x)(2\cos 2x) - \sin 2x\,(+2\sin 2x)}{(\cosh 2y - \cos 2x)^2}$$

$$\phi_2(z,0) = \left(\frac{\partial V}{\partial x}\right)_{(z,0)} = \frac{(1-\cos 2z)\,2\cos 2z - 2\sin^2 2z}{(1-\cos 2z)^2}$$

$$= \frac{(1-\cos 2z)\,2\cos 2z - 2(1-\cos^2 2z)}{(1-\cos 2z)^2}$$

$$= \frac{(1-\cos 2z)\,2\cos 2z - 2(1+\cos 2z)(1-\cos 2z)}{(1-\cos 2z)^2}$$

By Milne's Thomson method,

$$f(z) = \int [\phi_1(z,0) - i\,\phi_2(z,0)]\, dz$$

$$= \int [-\csc^2 z - i(0)]\, dz$$

$$= -\int \csc^2 z \, dz$$

$$f(z) = \cot z + C$$

4]. Find the analytic function $f(z) = u + iv$ where $u - v = e^x(\cos y - \sin y)$

Soln.

Let $f(z) = u + iv \rightarrow (1)$

$if(z) = iu - v \rightarrow (2)$

$(1)+(2) \Rightarrow (1+i)f(z) = u + iv + iu - v$

$(1+i)f(z) = (u-v) + i(u+v)$

$F(z) = U + iV$

Here $F(z) = (1+i)f(z)$

$U = u - v$

$V = u + v$

Given $U = u - v = e^x(\cos y - \sin y)$

$$\frac{\partial U}{\partial x} = e^x[\cos y - \sin y]$$

$$\phi_1(z,0) = \left(\frac{\partial U}{\partial x}\right)_{(z,0)} = e^z[1-0] = e^z$$

$$\left(\frac{\partial U}{\partial y}\right) = e^x[-\sin y - \cos y]$$

$$= -e^x[\sin y + \cos y]$$

$$\phi_2(z,0) = \left(\frac{\partial U}{\partial y}\right)_{(z,0)} = -e^z[0+1] = -e^z$$

3. Determine the analytic function whose real part is $\dfrac{\sin 2x}{\cosh 2y - \cos 2x}$

Soln.

Given $u = \dfrac{\sin 2x}{\cosh 2y - \cos 2x}$

$$\frac{\partial u}{\partial x} = \frac{(\cosh 2y - \cos 2x)\, 2\cos 2x - \sin 2x (2\sin 2x)}{(\cosh 2y - \cos 2x)^2}$$

$$\phi_1(z,0) = \left(\frac{\partial u}{\partial x}\right)_{(z,0)} = \frac{(1-\cos 2z)\,2\cos 2z - 2\sin^2 2z}{(1-\cos 2z)^2}$$

$$= \frac{(1-\cos 2z)\,2\cos 2z - 2(1-\cos^2 2z)}{(1-\cos 2z)^2}$$

$$= \frac{(1-\cos 2z)\,2\cos 2z - 2(1+\cos 2z)(1-\cos 2z)}{(1-\cos 2z)^2}$$

$$= \frac{2[\cos 2z - (1+\cos 2z)]}{(1-\cos 2z)}$$

$$= \frac{-2}{(1-\cos 2z)}$$

$$= \frac{-1}{\frac{1-\cos 2z}{2}} = \frac{-1}{\sin^2 z}$$

$$\phi_1(z,0) = -\csc^2 z$$

and $$\frac{\partial u}{\partial y} = \sin 2x \frac{\partial}{\partial y}\left[\frac{1}{\cosh 2y - \cos 2x}\right]$$

$$= \sin 2x \frac{-1}{[\cosh 2y - \cos 2x]^2}[2\sinh 2y - 0]$$

$$= \frac{-2\sin 2x \sinh 2y}{(\cosh 2y - \cos 2x)^2}$$

$$\phi_2(z,0) = \left(\frac{\partial u}{\partial y}\right)_{(z,0)} = \frac{-2\sin 2z\,(0)}{(1-\cos 2z)^2} = 0$$

$$= \frac{2\cos 2x - 2(1+\cos 2x)}{1-\cos 2x}$$

$$= \frac{2[\cos 2x - 1 - \cos 2x]}{1-\cos 2x}$$

$$= \frac{-2}{1-\cos 2x} = \frac{-1}{\frac{1-\cos 2x}{2}} = \frac{-1}{\sin^2 x}$$

$$\phi_2(x,0) = -\csc^2 x$$

$$\frac{\partial v}{\partial y} = \sin 2x \frac{-1}{(\cosh 2y - \cos 2x)^2}[2\sinh 2y - 0]$$

$$= \frac{-2\sin 2x \sinh 2y}{(\cosh 2y - \cos 2x)^2}$$

$$\phi_1(x,0) = \left(\frac{\partial v}{\partial y}\right)_{(x,0)} = 0$$

By Milne's Thomson method,

$$F(z) = \int [\phi_1(z,0) + i\phi_2(z,0)]\, dz$$

$$= \int [0 + i(-\csc^2 z)]\, dz$$

$$= -i\int \csc^2 z\, dz$$

$$F(z) = i\cot z + c$$

$$(1+i)f(z) = i\cot z + c$$

$$f(z) = \frac{i}{1+i}\cot z + \frac{c}{1+i}$$

$$= \frac{i(1-i)}{(1+i)(1-i)}\cot z + C$$

$$= \frac{i - i^2}{1^2 - i^2}\cot z + C$$

$$f(z) = \frac{(1+i)}{2}\cot z + C$$