Standard Transformation:

TYPE-1

I. Translation $[w = c + z]$

problem:

1. What is the region of the w-plane into which the rectangular region in the z-plane bounded by the lines $x=0, y=0, x=1$ and $y=2$ is mapped under the transformation $w = z + (2-i)$?

Soln.

Given $w = z + (2-i)$

$u + iv = x + iy + (2-i)$

$u + iv = (x+2) + i(y-1)$

Equating real & imaginary parts,

$u = x + 2$ and $v = y - 1$

Boundary lines   Transformed boundary lines

$x = 0$ $u = 2$

$y = 0$ $v = -1$
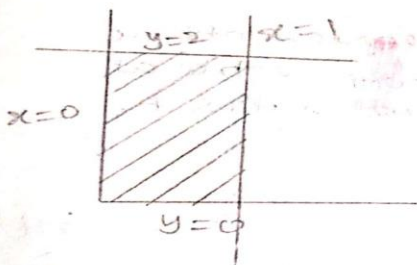
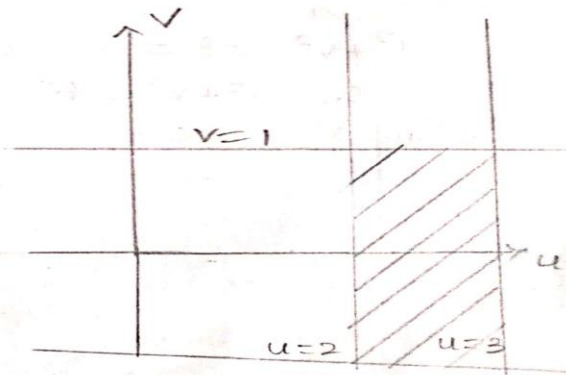$x = 1$ $u = 3$

$y = 2$ $v = 1$

z-plane

w-plane

Hence the lines $x=0, y=0, x=1$ & $y=2$ in z-plane are transformed into $u=2, v=-1, u=3$ and $v=1$ respectively in w-plane

$Hw |z-2) = 3$ $w = z + (3-2i)$

TYPE 2 :

Magnification and Rotation $[\;\; w = cz\;]$

1. Find the image of circle $|z| = 3$ under the transformation $w = 2z$.

Soln:

     Given $w = 2z$

$$u + iv = 2(x + iy)$$
$$u + iv = 2x + i2y$$

$$\Rightarrow u = 2x \quad \text{and} \quad v = 2y$$

$$x = \frac{u}{2} \qquad\qquad y = \frac{v}{2}$$

To find the image of $|z| = 3$.

$$|x + iy| = 3$$
$$\sqrt{x^2 + y^2} = 3$$
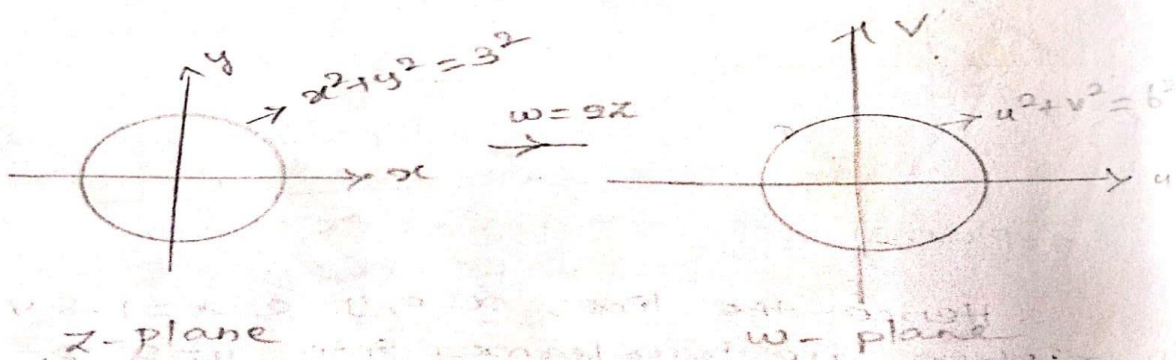$$x^2 + y^2 = 3^2$$
$$x^2 + y^2 - 9 = 0$$
$$\left(\frac{u}{2}\right)^2 + \left(\frac{v}{2}\right)^2 - 9 = 0$$
$$\frac{u^2 + v^2 - 36}{4} = 0$$
$$u^2 + v^2 - 36 = 0$$

i.e., $u^2 + v^2 = 6^2$ which represents a circle with centre at origin and radius 6.

z-plane   w-plane

2]. Find the image of the region $y > 1$ under $\omega = (1-i)z$

Soln.

Given $\omega = (1-i)z$

$$u+iv = (1-i)(x+iy)$$
$$= x+iy-ix+y$$
$$u+iv = (x+y)+i(y-x)$$
$$\Rightarrow u = x+y \quad \text{and} \quad v = y-x$$

Now,

$$u+v = x+y+y-x \qquad u-v = x+y-y+x$$
$$u+v = 2y \qquad\qquad\quad u-v = 2x$$
$$y = \frac{u+v}{2} \qquad\qquad\quad x = \frac{u-v}{2}$$

To find the image of $y > 1$

$$\text{i.e., } \frac{u+v}{2} > 1$$

$$\Rightarrow u+v > 2$$

$$\omega = (1-i)z$$

z - plane   w - plane
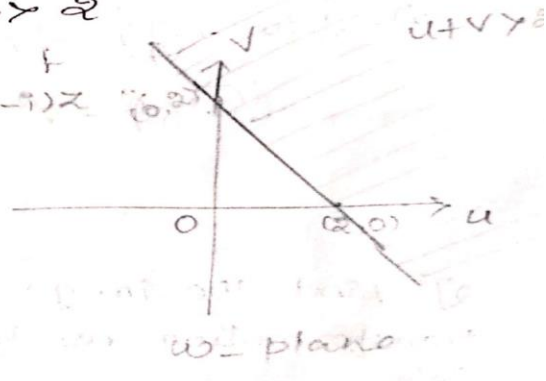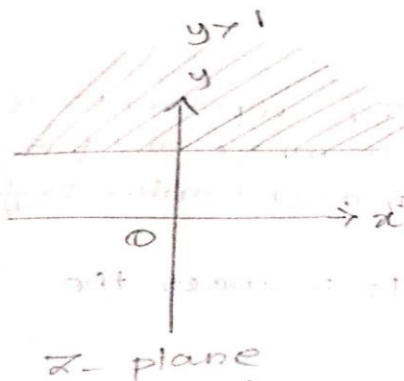
Type 3:

Reversion and Reflection $\left[\omega = \frac{1}{z} \right]$

1. Find the image of $x = 2$ under the transformation $\omega = \frac{1}{z}$.

Soln.

Given $\omega = \frac{1}{z}$

$$\bar{z} = \frac{1}{\omega}$$

$$x + iy = \frac{1}{u + iv} \times \frac{u - iv}{u - iv}$$

$$= \frac{u - iv}{u^2 + v^2}$$

$$x + iy = \frac{u}{u^2 + v^2} - i \cdot \frac{v}{u^2 + v^2}$$

$$\Rightarrow x = \frac{u}{u^2 + v^2} \quad \text{and} \quad y = -\frac{v}{u^2 + v^2}$$

$$\hookrightarrow (1) \qquad \qquad \hookrightarrow (2)$$

when $x = 2$

$$(1) \Rightarrow \quad 2 = \frac{u}{u^2 + v^2}$$

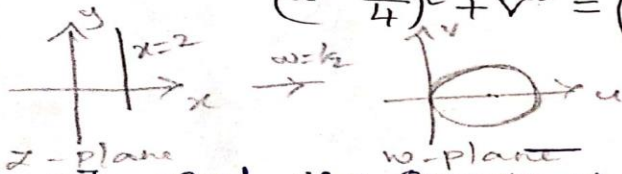$$2(u^2 + v^2) = u$$

$$u^2 + v^2 - \frac{1}{2}u = 0$$

$$u^2 - \frac{1}{2}(u) + \left(\frac{1}{4}\right)^2 - \left(\frac{1}{4}\right)^2 + v^2 = 0$$

$$2ub = \frac{1}{2}u$$

$$b = \frac{1}{4}$$

$$\left(u - \frac{1}{4}\right)^2 + v^2 = \left(\frac{1}{4}\right)^2$$

which represents a circle whose centre is $(\frac{1}{4}, 0)$ and radius is $\frac{1}{4}$

z - plane   w - plane

2]. Find the image of $|z - 2i| = 2$ under the transformation $w = \frac{1}{z}$.

Soln.

Given $w = \frac{1}{z}$

$$z = \frac{1}{\omega}$$

$$x + iy = \frac{1}{u + iv} \times \frac{u - iv}{u - iv}$$

$$x + iy = \frac{u - iv}{u^2 + v^2}$$

$$\Rightarrow x = \frac{u}{u^2+v^2} \quad \text{and} \quad y = \frac{-v}{u^2+v^2}$$

To find the image of $|x-2i| = 2$

$$|x+iy-2i| = 2$$

$$|x+i(y-2)| = 2$$

$$x^2+(y-2)^2 = 2^2 \quad \text{which is a circle with centre } (0,2) \text{ & radius } 2.$$

$$x^2+y^2+4-4y-4 = 0$$

$$x^2+y^2-4y = 0$$

$$\left(\frac{u}{u^2+v^2}\right)^2 + \left(\frac{-v}{u^2+v^2}\right)^2 - 4\left(\frac{-v}{u^2+v^2}\right) = 0$$

$$\frac{u^2}{(u^2+v^2)^2} + \frac{v^2}{(u^2+v^2)^2} + \frac{4v}{u^2+v^2} = 0$$

$$\frac{u^2+v^2+4v(u^2+v^2)}{(u^2+v^2)^2} = 0$$

$$(u^2+v^2)+4v(u^2+v^2) = 0$$
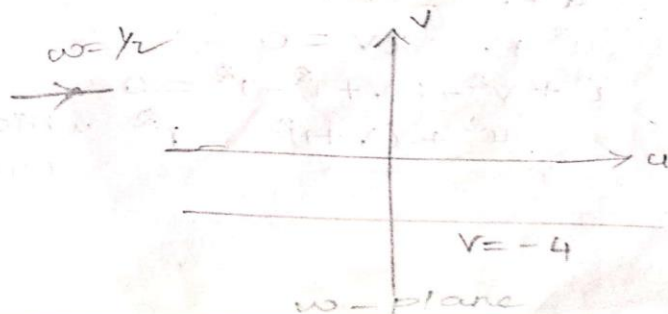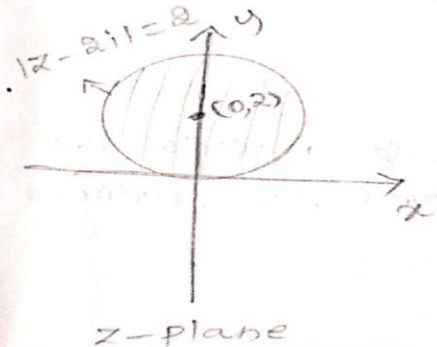
$$\frac{(u^2+v^2)(1+4v)}{(u^2+v^2)^2} = 0$$

$$\frac{1+4v}{u^2+v^2} = 0$$

$$1+4v = 0$$

$$4v = -1$$

$$v = -\frac{1}{4} \quad \text{which is a straight line}$$

z-plane   w-plane

2. Find the image of infinite strip

i). $\frac{1}{4} < y < \frac{1}{2}$ ii). $0 < y < \frac{1}{2}$ under the transformation $\omega = \frac{1}{z}$.

Soln.

Given $\omega = \frac{1}{z}$

$$z = \frac{1}{\omega}$$

$$x+iy = \frac{1}{u+iv} \times \frac{u-iv}{u-iv}$$

$$x+iy = \frac{u-iv}{u^2+v^2}$$

$$x = \frac{u}{u^2+v^2} \quad \text{and} \quad y = \frac{-v}{u^2+v^2}$$

i). $\frac{1}{4} < y < \frac{1}{2}$

when $y = \frac{1}{4}$, $\frac{-v}{u^2+v^2} = \frac{1}{4}$

$$u^2+v^2 = -4v$$

$$u^2+v^2+4v = 0$$

$$u^2+v^2+4v+2^2-2^2 = 0$$

$$u^2+(v+2)^2 = 4 \quad \text{which is a circle}$$
with centre $(0,-2)$ & radius 2.

when $y = \frac{1}{2}$,

$$\frac{1}{2} = \frac{-v}{u^2+v^2}$$

$$u^2+v^2 = -2v$$

$$u^2+v^2+2v = 0$$

$$u^2+v^2+2v+1^2-1^2 = 0$$

$$u^2+(v+1)^2 = 1^2 \quad \text{which is a circle with}$$
centre $(0,-1)$ & Radius 1

Hence the infinitestrip $\frac{y}{4} < y < \frac{1}{2}$ in z-plane are transformed into region b/w the circle in w-plane.
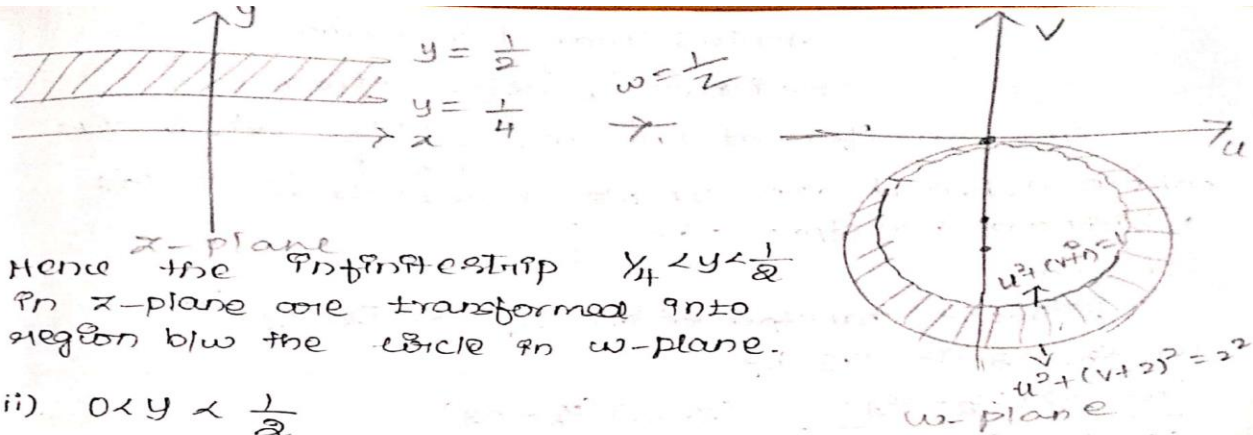
ii) $0 < y < \frac{1}{2}$

when $y = 0$

$$\frac{-v}{u^2+v^2} = 0$$
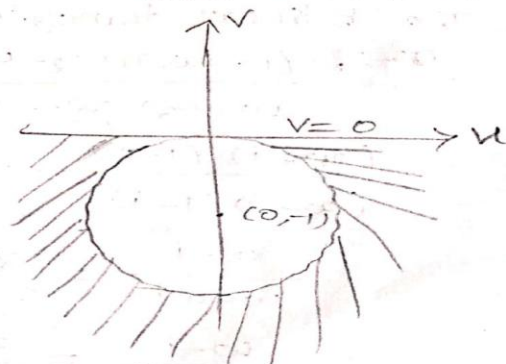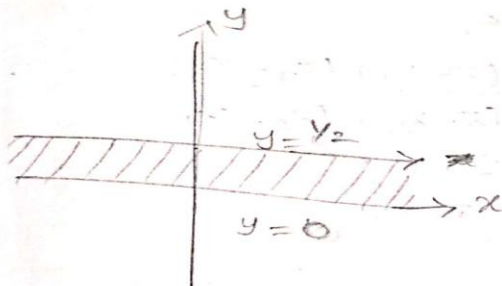
$$-v = 0$$

$$v = 0$$

$y = \frac{1}{2}$

$$\frac{-v}{u^2+v^2} = \frac{1}{2}$$

$$u^2 + v^2 = -2v$$

$$u^2 + v^2 + 2v + 1^2 - 1^2 = 0$$

$$u^2 + (v+1)^2 = 1^2 \text{ which represents}$$

a circle with $C(0,-1)$ & $R(1)$

Hence the infinitestrip $0 < y < \frac{1}{2}$ in z-plane are transformed into the region outside the circle $u^2 + (v+1)^2 = 1$ and lower half of the w-plane.