



## UNIT - 3

### Introduction

If  $x$  &  $y$  are real numbers then  $z = x + iy$  is called a complex number where  $x$  is called real part of  $z$ ,  $y$  is called imaginary part of  $z$  and the value of  $i$  is  $\sqrt{-1}$ . The complex number  $x - iy$  is called the complex conjugate of  $z$  and it is denoted by  $\bar{z}$ . i.e)  $\bar{z} = x - iy$

Note :

1.  $|z| = \sqrt{x^2 + y^2}$

2.  $|z^2| = z\bar{z}$

3.  $z\bar{z} = x^2 + y^2 = r^2$

4.  $|\bar{z}| = |z|$

5. Real part of  $z = \frac{z + \bar{z}}{2}$

6. Imaginary part of  $z = \frac{z - \bar{z}}{2}$

7.  $z = re^{i\theta}$  is called polar form of  $z$

Function of complex variable

$w = f(z) = u(x, y) + iv(x, y)$  where  $u(x, y)$  &  $v(x, y)$  are real variables.

Analytic function

A function is said to be analytic at a point if its derivative exists not only at that point but also some neighbourhood of that point



1. Show that the function  $f(z) = \bar{z}$  is nowhere differentiable.

Soln.

$$\text{Given } f(z) = \bar{z} = x - iy$$

$$u + iv = x - iy$$

$$\Rightarrow u = x \text{ and } v = -y$$

$$u_x = 1 \quad v_x = 0$$

$$u_y = 0 \quad v_y = -1$$

Here  $u_x \neq v_y$  and  $u_y \neq -v_x$

Hence C-R eqns are not satisfied.

$\Rightarrow f(z) = \bar{z}$  is not differentiable anywhere (or) nowhere differentiable.

2. Determine whether the function  $2xy + i(x^2 - y^2)$  is analytic or not.

Soln.

$$\text{Let } f(z) = 2xy + i(x^2 - y^2)$$

$$u + iv = 2xy + i(x^2 - y^2)$$

$$\Rightarrow u = 2xy \text{ and } v = x^2 - y^2$$

$$u_x = 2y \quad v_x = 2x$$

$$u_y = 2x \quad v_y = -2y$$

$$\Rightarrow u_x \neq v_y \text{ and } u_y \neq -v_x$$

C.R eqns are not satisfied.

Hence  $f(z)$  is not an analytic function.

3. Let  $f(z) = z^3$  be analytic. Justify.

Soln.

$$\text{Let } f(z) = z^3$$

$$u + iv = (x + iy)^3$$

$$= x^3 + 3x^2(iy) + 3x(iy)^2 + (iy)^3$$

$$= x^3 + i3x^2y - 3xy^2 - iy^3$$



$$u+iv = [x^3 - 3xy^2] + i [3x^2y + y^3]$$

$$\Rightarrow u = x^3 - 3xy^2 \quad \text{and} \quad v = 3x^2y + y^3$$

$$u_x = 3x^2 - 3y^2$$

$$v_x = 6xy$$

$$u_y = -6xy$$

$$v_y = 3x^2 + 3y^2$$

$$\Rightarrow u_x = v_y \quad \text{and} \quad u_y = -v_x$$

CR eqns are satisfied.

Hence  $f(z)$  is analytic.

A. Find the constants  $a, b, c$  if  $f(z) = x + ay + i(bx + cy)$  is analytic.

Soln.

$$\text{Let } f(z) = x + ay + i(bx + cy)$$

$$u+iv = x + ay + i(bx + cy)$$

$$\text{Here } u = x + ay \quad \text{and} \quad v = bx + cy$$

$$u_x = 1$$

$$v_x = b$$

$$u_y = a$$

$$v_y = c$$

Since  $f(z)$  is analytic.

$$\Rightarrow u_x = v_y \quad \text{and} \quad u_y = -v_x$$

$$1 = c$$

$$a = -b$$

$$\therefore a = -b \quad \text{and} \quad c = 1.$$